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Title: Causality versus True Concurrency in the Setting of Real-Time Models

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Abstract: The contribution of the paper is to clarify connections between real-time models of concurrency. In particular, we defined a category of timed causal trees and investigated how it relates to other categories of timed models. Moreover, using a larger model called timed event trees, we constructed an adjunction from the category of timed causal trees to the category of timed event structures. Thereby we showed that timed causal trees are more trivial than timed event structures because they reflect only one aspect of true concurrency, causality, and they apply causality without a notion of event. On the other hand, the first model is more expressive than the latter in that possible runs of a timed causal tree can be defined in terms of a tree without restrictions, but the set of the possible runs of any event structure must be closed under the shuffling of concurrent transitions.

Keywords: real-time models, true concurrency, causality, relations, unification, category theory

1. Introduction. In recent decades, category theoretical approaches have been actively used for the specification and investigation of concurrent systems and processes. We will mention just one example, which is directly related to the concurrency theory. The category theory has helped us to classify and unify various models for concurrency and has provided an abstract language for expressing relationships between seemingly very different models. The basic goal is to formulate the fact that one model is more expressive than another in terms of an 'embedding' or coreflection (reflection) — the category theoretical notion defined as an adjunction, in which the unit (counit) is an isomorphism. In the setting of this approach, models are represented as categories: each model is equipped with a notion of morphism that shows how one model instance can be simulated by another. Moreover, the existence of (co)reflection between models allows us to translate concepts and properties from one model to another.

At present, the concurrency theory has a great variety of formal models that can be classified based on different principles. For example, concurrent models are split to interleaving models and true concurrent models. For interleaving models, such as synchronization trees, causal trees and transition systems, the concurrency is simulated by a sequence of actions. For true concurrent models, such as event structures, transition systems with independence, labelled asynchronous transition systems, causal trees and Petri nets, the concurrency is modelled implicitly through the relation of independence.

In [3, 5] Winskel, Nielsen and Joyal have applied the category theory to unify the many models for concurrency and to establish the relationships between them. They have shown that the categories of such models as synchronization trees, transition systems, event structures, transition systems with independence and asynchronous transition systems are related by coreflections. In particular, they have found out the following facts. Intuitively, synchronization trees are transition systems with no cyclic behaviour. Moreover, a synchronization tree may be transferred to a special kind of an event structure with an empty independence relation. The transition systems may be regarded as transition systems with independence in which the independence relation is empty. An event structure may be translated to a special type of a transition system with independence. Finally, transition systems with independence may be considered as asynchronous transition systems, which have at most one transition with the same label between two same states. Later Nielsen and Winskel proved that there exists a coreflection between Petri nets and asynchronous transition systems (see [4]). In [2] Fröschle and Lasota integrated a new model, the causal trees of Darondeau and Degano, into Winskel and Nielsen's framework. Also they have shown that there is an adjunction from causal trees to event structures. Causal trees are some variant of synchronization trees with enriched action labels that supply information about which transitions causally depend on each other. Thereby, they reflect the only one aspect of true concurrency, causality. On the other hand, there is one aspect in which event structures are less expressive than causal trees: their notion of run is induced abstractly by the consistency and causal dependency relation. In particular, this means the set of runs of any event structure is closed under the shuffling of concurrent transitions.

More recently, great efforts have been made to develop formal methods for real-time systems. These are systems whose correctness depends crucially upon real-time considerations. As a result, time extensions of concurrent models such as timed automata, times synchronization trees, timed transition systems, timed event structures, and timed Petri nets have appeared and have been investigated. However, only a few examples of the category theoretical classification for timed models are described in literature.

The contribution of the paper is to show the applicability of the general categorical framework proposed by Winskel and Nielsen and to clarify connections between real-time models of concurrency. In particular, we defined categories for such models as timed transition systems, timed synchronization trees, timed causal trees and timed event structure, and investigated how they relate with each other. Moreover, using a larger model called timed event trees we showed the existence of an adjunction from the category of timed causal trees to the category of timed event structures.

The rest of the paper is organized as follows. The basic notions and notations of the category theory are introduced in Section 2. In the next section, we define categories for timed extensions of concurrent models and establish some of their properties. Five subsections of Section 3 describe five different models: timed transition systems, timed synchronization trees, timed causal trees, timed event structures and timed event trees. Relations between timed models for concurrency are introduced in Section 4, which consists of five subsections. In the first subsection a coreflection between the category of timed causal trees and the category of timed synchronization trees is exhibited. The existence of a coreflection between the category of timed causal trees and the category of timed event trees is shown in Subsection 4.2. The next subsection proves the existence of a reflection between the category of timed event structures and the category of timed event trees. In the fourth subsection, the construction of a coreflection between the category of timed event structures and the category of timed synchronization trees is described. Each of above subsections consists of definitions of two functors between two certain categories, some useful propositions and the main theorem, which asserts the existence of (co)reflection between the categories. Subsection 4.5 recapitulates the obtained results. Section 5 is the conclusion of the paper.

2. Basics of the Category Theory. In this section we will briefly recall some basic notions and notations from the category theory [1]. Let us start with the definition of a category.

Definition 1. A category $\mathcal{C}$ *consists of the following:*

- *a class* $|\mathcal{C}|$*, whose elements will be called "objects of the category";*
- *for every pair* A*,* B *of objects, a set* $\mathcal{C}(A,B)$*, whose elements will be called "morphisms" or "arrows" from* A *to* B*;*
- *for every triple* A*,* B*,* C *of objects, a composition law* $\mathcal{C}(A,B)\times\mathcal{C}(B,C)\longrightarrow\mathcal{C}(A,C)$*. The composite of the pair* (f,g) *will be written* $g\circ f$ *or just* gf*;*
- *for every object* A*, a morphism* $I_A\in\mathcal{C}(A,A)$*, called the identity on* A*.*

These data are subject to the following axioms.

- Associativity axiom*: given morphisms* $f\in\mathcal{C}(A,B)$*,* $g\in\mathcal{C}(B,C)$*,* $h\in\mathcal{C}(C,D)$ *the following equality holds:* $h\circ(g\circ f)=(h\circ g)\circ f$*;*
- Identity axiom*: given morphisms* $f\in\mathcal{C}(A,B)$*,* $g\in\mathcal{C}(B,C)$*, the following equalities hold:* $1_B\circ f=f$*,* $g\circ 1_B=g$*.*

Now we adduce the notion of a functor (or a "homomorphism of categories") with some of their properties.

Definition 2. A functor F *from a category* $\mathcal{C}$ *to a category* $\mathcal{D}$ *consists of the following:*

- *a mapping* $|\mathcal{C}|\longrightarrow|\mathcal{D}|$ *between the classes of objects of* $\mathcal{C}$ *and* $\mathcal{D}$*; the image of* $A\in\mathcal{C}$ *is written* $F(A)$ *or just* FA*;*

- *for every pair of objects* A, A' *of* $\mathcal{C}$, *a mapping* $\mathcal{C}(A, A') \longrightarrow \mathcal{D}(FA, FA')$; *the image of* $f \in \mathcal{C}(A, A')$ *is written* $F(f)$ *or just* Ff.

These data are subject to the following axioms:

- *for every pair of morphisms* $f \in \mathcal{C}(A, A')$, $g \in \mathcal{C}(A', A'')$ $F(g \circ f) = F(g) \circ F(f)$;
- *for every object* $A \in \mathcal{C}$ $F(1_A) = 1_{FA}$.

Definition 3. *Consider a functor* $F : \mathcal{C} \to \mathcal{D}$ *and for every pair of objects* $A, A' \in \mathcal{C}$, *the mapping* $\mathcal{C}(A, A') \to \mathcal{D}(FA, FA')$, $f \mapsto Ff$.

- *The functor* F *is faithful when the above mentioned mappings are injective for all* A, A';
- *The functor* F *is full when the above mentioned mappings are surjective for all* A, A';
- *The functor* F *is full and faithful when the above mentioned mappings are bijective for all* A, A';
- *The functor* F *is an isomorphism of categories when it is full and faithful and induces a bijection* $|\mathcal{C}| \longrightarrow |\mathcal{D}|$ *on the classes of objects.*

There is a notion of natural transformations in the category theory, which is an adaptation of the notion of a "homotopy" between two continuous functions from one space to another.

Definition 4. *Consider two functors* $F, G : \mathcal{C} \to \mathcal{D}$ *from a category* $\mathcal{C}$ *to a category* $\mathcal{D}$. A natural transformation $\alpha : F \Rightarrow G$ *from* F *to* G *is a class of morphisms* $(\alpha_A : FA \to GA)_{A \in \mathcal{C}}$ *of* $\mathcal{D}$ *indexed by the objects of* $\mathcal{C}$ *and such that for every morphism* $f : A \to A'$ *in* $\mathcal{C}$, $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$.

One of the basic conceptions of the category theory is a notion of adjoint functors. There are various definitions for adjoint functors. Their equivalence is elementary but not at all trivial. We will use the definitions via reflections and coreflections along functors.

Definition 5. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor and* B *an object of* $\mathcal{D}$. A reflection of B along F *is a pair* (R_B, η_B) *where* R_B *is an object of* $\mathcal{C}$, $\eta_B : B \to F(R_B)$ *is a morphism of* $\mathcal{D}$, *and if* $A \in |\mathcal{C}|$ *is an object of* $\mathcal{C}$ *and* $b : B \to F(A)$ *is a morphism of* $\mathcal{D}$, *then there exists a unique morphism* $a : R_B \to A$ *in* $\mathcal{C}$ *such that* $F(a) \circ \eta_B = b$.

Definition 6. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor and* B *an object of* $\mathcal{D}$. A coreflection of B along F *is a pair* (R_B, ϵ_B) *where* R_B *is an object of* $\mathcal{C}$, $\epsilon_B : F(R_B) \to B$ *is a morphism of* $\mathcal{D}$, *and if* $A \in |\mathcal{C}|$ *is an object of* $\mathcal{C}$ *and* $b : F(A) \to B$ *is a morphism of* $\mathcal{D}$, *then there exists a unique morphism* $a : A \to R_B$ *in* $\mathcal{C}$ *such that* $\epsilon_B \circ F(a) = b$.

Definition 7. *A functor $R : \mathcal{D} \to \mathcal{C}$ is* left adjoint *to the functor $F : \mathcal{C} \to \mathcal{D}$ (and F is* right adjoint *to R) when there exists a natural transformation $\eta : 1_{\mathcal{D}} \Rightarrow F \circ R$, called the unit of the adjunction, such that for every $B \in \mathcal{D}$, a pair (RB, η_B) is a reflection of B along F.*

Definition 8. *A functor $R : \mathcal{D} \to \mathcal{C}$ is* right adjoint *to the functor $F : \mathcal{C} \to \mathcal{D}$ (and F is* left adjoint *to R) when there exists a natural transformation $\epsilon : F \circ R \Rightarrow 1_{\mathcal{D}}$, called the counit of the adjunction, such that for every $B \in \mathcal{D}$, a pair (RB, ϵ_B) is a coreflection of B along F.*

We will call an adjunction in which the unit (the counit) is a natural isomorphism as *a coreflection* (*a reflection*).

3. Models for Concurrency. In this section we study the timed extensions of five different concurrent models. Four of them are well-known interleaving and true concurrency models, and the fifth one is called event trees and embeds causal trees as well as event structures. Event trees are like event structures because causality and concurrency are event-based, global notions. They are like causal trees because their possible runs are specified explicitly by a tree.

We start by introducing of timed variants of the models, and then we define categories for them.

3.1. Timed Transition Systems. Let $\mathbf{R}$ be a set of non-negative reals and L be a finite alphabet of actions. Consider the definition of timed transition systems.

Definition 9. *A* timed transition system *$\mathcal{T}$* over an alphabet *L is a tuple (S, s_{in}, L, T), where S is a set of states and s_{in} is the initial state, $T \subseteq S \times L \times \mathbf{R} \times \mathbf{R} \times S$ is a set of transitions such that for all $(s, \sigma, eot, lot, s') \in T$ we have $eot \le lot$. We will write $s \xrightarrow[eot,\ lot]{\sigma} s'$ to denote a transition $(s, \sigma, eot, lot, s')$.*

Let us define the behaviour of timed transition systems.

Definition 10. *Let $\mathcal{T}$ be a timed transition system over L.*

A configuration *of $\mathcal{T}$ is a pair $\langle s, \nu \rangle$, where s is a state and ν is a current global time moment.*

A run *of $\mathcal{T}$ is a sequence $\gamma = \langle s_0, \nu_0 \rangle \stackrel{\sigma_1}{\to} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \stackrel{\sigma_n}{\to} \langle s_n, \nu_n \rangle$ such that $\nu_1 \le \ldots \le \nu_n$ and for all $1 \le i \le n$ there is a transition $s_{i-1} \xrightarrow[eot_i,\ lot_i]{\sigma_i} s_i$ such that $eot_i \le \nu_i \le lot_i$. Here, $s_0 = s_{in}$ and ν_0 is defined to be 0.*

We are now ready to introduce the category of timed transition systems.

Definition 11. *Given timed transition systems $\mathcal{T} = (S,\ s_{in},\ L,\ T)$ and $\mathcal{T}' = (S',\ s'_{in},\ L',\ T')$, a pair (μ, λ) is a* morphism *between $\mathcal{T}$ and $\mathcal{T}'$, if $\mu : S \to S'$ and $\lambda : L \to L'$ are functions such that $\mu(s_{in}) = s'_{in}$, and if $(s, \sigma, eot, lot, s') \in T$, then $(\mu(s), \lambda(\sigma), eot', lot', \mu(s')) \in T'$ for some real numbers eot' and lot' such that $eot' \le eot$ and $lot \le lot'$.*

Timed transition systems and morphisms between them form a category of timed transition systems, **TTS**, in which the composition of two morphisms $(\mu,\lambda):\mathcal{T}\to\mathcal{T}'$ and $(\mu',\lambda'):\mathcal{T}'\to\mathcal{T}''$ is defined as $(\mu',\lambda')\circ(\mu,\lambda) := (\mu'\circ\mu, \lambda'\circ\lambda)$, and the identity morphism is a pair of the identity functions.

Lemma 1. *Given a morphism* $(\mu,\lambda):\mathcal{T}\to\mathcal{T}'$ *of* **TTS**, *if* $\langle s_0,\nu_0\rangle \overset{\sigma_1}{\to} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \overset{\sigma_n}{\to} \langle s_n,\nu_n\rangle$ *is a run of* $\mathcal{T}$ *then* $\langle \mu(s_0),\nu_0\rangle \overset{\lambda(\sigma_1)}{\to} \langle \mu(s_1),\nu_1\rangle \ldots \langle \mu(s_{n-1}),\nu_{n-1}\rangle \overset{\lambda(\sigma_n)}{\to} \langle \mu(s_n),\nu_n\rangle$ *will be a run of* $\mathcal{T}'$.

3.2. Timed Synchronization Trees. Now we contemplate the definition of timed synchronization trees.

Definition 12. *A* timed synchronization tree $\mathcal{S}$ *is a timed transition system* (S, s_{in}, L, T) *such that*

(i) for all $s \in S$ *there exists a sequence* $s_{in} \xrightarrow[eot_1,\ lot_1]{\sigma_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{\sigma_k} s_k$ $(k \geq 0)$ *such that* $s = s_k$;

(ii) for all sequence $s_0 \xrightarrow[eot_1,\ lot_1]{\sigma_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{\sigma_k} s_k$ $(k \geq 0)$ *it holds if* $s_0 = s_k$ *then* $k = 0$;

(iii) if $s' \xrightarrow[eot,\ lot]{\sigma} s$ *and* $s'' \xrightarrow[eot',\ lot']{\sigma'} s$, *then* $s' = s''$, $\sigma = \sigma'$, $eot = eot'$ *and* $lot = lot'$.

Write **TST** for the full subcategory of timed synchronization trees in **TTS**.

3.3. Timed Causal Trees. In this subsection we introduce the timed extension of causal trees, which are a generalization of synchronization trees.

Definition 13. *A* timed causal tree $\mathcal{C}$ *is a tuple* $(S, s_{in}, L, T, <)$ *where* $(S,\ s_{in},\ L,\ T)$ *is a timed synchronization tree and* $< \subseteq T\times T$, *the* causal dependency relation, *is a strict order such that for all transitions* $(s,\ \sigma,\ eot,\ lot,\ s')$ *and* $(s'',\ \sigma',\ eot',\ lot',\ s''')$ *of* $\mathcal{C}$ *if* $(s,\sigma,eot,lot,s') < (s'',\sigma',eot',lot',s''')$, *then there exists a sequence* $s' \xrightarrow[eot_1,\ lot_1]{\sigma_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{\sigma_k} s''$ *for some* $k \geq 0$.

We will say that two transitions (s,σ,eot,lot,s') and $(s'',\sigma',eot',lot',s''')$ of $\mathcal{C}$ are *consistent* (denoted $(s,\sigma,eot,lot,s')\ Con\ (s'',\sigma',eot',lot',s''')$) iff either $(s,\ \sigma,\ eot,\ lot,\ s') = (s'',\ \sigma',\ eot',\ lot',\ s''')$ or there exists a sequence $s_0 \xrightarrow[eot_1,\ lot_1]{\sigma_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{\sigma_k} s_k$ $(k \geq 0)$ such that $(s' = s_0 \wedge s'' = s_k)$ or $(s''' = s_0 \wedge s = s_k)$. A *run* of $\mathcal{C} = (S, s_{in}, L, T, <)$ is a sequence $\gamma = \langle s_0,\nu_0\rangle \underset{K_1}{\overset{\sigma_1}{\to}} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \underset{K_n}{\overset{\sigma_n}{\to}} \langle s_n,\nu_n\rangle$ such that $\langle s_0,\nu_0\rangle \overset{\sigma_1}{\to} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \overset{\sigma_n}{\to} \langle s_n,\nu_n\rangle$ is a run of (S, s_{in}, L, T) and $K_i = \{j \mid 1 \leq j \leq i,\ (s_{j-1},\sigma_j,eot_j,lot_j,s_j) < (s_{i-1},\sigma_i,eot_i,lot_i,s_i)\}$ for all $1 \leq i \leq n$.

We are ready to equip timed causal trees with a notion of morphism and thus define a category of timed causal trees.

Definition 14. *Given timed causal trees* $\mathcal{C} = (S,\ s_0,\ L,\ T, <)$ *and* $\mathcal{C}' = (S',\ s_0',\ L',\ T',\ <')$*, a pair* (μ,λ) *is a* morphism *between* $\mathcal{C}$ *and* $\mathcal{C}'$*, if* (μ,λ) *is a morphism between timed synchronization trees* $(S,\ s_0,\ L,\ T)$ *and* $(S',\ s_0',\ L',\ T')$ *and for all transitions* (s,σ,eot,lot,s_1) *and* $(s_2,\sigma_1,eot_1,lot_1,s_3)$ *of* $\mathcal{C}$*, if* (s,σ,eot,lot,s_1) *Con* $(s_2,\sigma_1,eot_1,lot_1,s_3)$ *and* $(\mu(s),\ \lambda(\sigma),\ eot',\ lot',\ \mu(s_1)) <' (\mu(s_2),\ \lambda(\sigma_1),\ eot_1',\ lot_1',\ \mu(s_3))$ *for some* eot'*,* lot'*,* eot_1'*,* $lot_1' \in \mathbf{R}$ *such that* $eot' \leq eot$*,* $lot \leq lot'$*,* $eot_1' \leq eot_1$ *and* $lot_1 \leq lot_1'$*, then* $(s,\sigma,eot,lot,s_1) < (s_2,\sigma_1,eot_1,lot_1,s_3)$*.*

Timed causal trees and their morphisms form a category of timed causal trees, **TCT**.

Lemma 2. *Given a morphism* $(\mu,\lambda) : \mathcal{C} \to \mathcal{C}'$ *of* **TCT***, if* $\gamma = \langle s_0,\nu_0\rangle \underset{K_1}{\overset{\sigma_1}{\to}} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \underset{K_n}{\overset{\sigma_n}{\to}} \langle s_n,\nu_n\rangle$ *is a run of* $\mathcal{C}$*, then* $\gamma' = \langle \mu(s_0),\nu_0\rangle \underset{K_1'}{\overset{\lambda(\sigma_1)}{\to}} \langle \mu(s_1),\nu_1\rangle \ldots \langle \mu(s_{n-1}),\nu_{n-1}\rangle \underset{K_n'}{\overset{\lambda(\sigma_n)}{\to}} \langle \mu(s_n),\nu_n\rangle$ *will be a run of* $\mathcal{C}'$ *for some* $K_1',\ldots,K_n'$ *such that* $K_i' \subseteq K_i$ *for all* $1 \leq i \leq n$*.*

3.4. Timed Event Structures. This subsection is dedicated to the most popular true concurrency model — timed event structures. Let us first give the definition of this model.

Definition 15. *A* timed event structure *is a tuple* $\mathcal{E} = (E,\ <,\ Con,\ L,\ l,\ Eot,\ Lot)$*, where* E *is a set of events;* $< \subseteq E \times E$ *is a strict order (the* causality relation*), satisfying the* principle of finite causes*:* $\forall e \in E \diamond e\downarrow = \{e' \in E \mid e' < e\}$ *is finite;* $Con \subseteq 2^E$ *(the* consistency relation*) consists of finite subsets of events which can occur together in a run, satisfying the following* principles*:* $\forall e \in E \diamond \{e\} \in Con$*;* $Y \subseteq X \in Con \Rightarrow Y \in Con$ *and* $X \in Con \wedge e < e' \in X \ \Rightarrow X \cup \{e\} \in Con$*;* L *is a set of actions;* $l : E \to L$ *is a labelling function and* $Eot,\ Lot : E \to \mathbf{R}$ *are functions of the earliest and the latest occurrence times of events, satisfying the following:* $Eot(e) \leq Lot(e)$ *for all* $e \in E$*.*

Let $C \subseteq E$. Then C is *left-closed* iff $\forall e,e' \in E \diamond e \in C \ \wedge\ e' < e \Rightarrow e' \in C$; C is *consistent* iff $C \in Con$; C is a *configuration of* $\mathcal{E}$ iff C is left-closed and consistent. Let $\mathbf{C}(\mathcal{E})$ denote the set of all finite configurations of $\mathcal{E}$.

An execution of a timed event structure is a *timed configuration* which consists of a configuration and a timing function recording global time moments at which events occur and satisfies some additional requirements. Let $\mathcal{E} = (E,\ <,\ Con,\ L,\ l,\ Eot,\ Lot)$ be a timed event structure, $C \in \mathbf{C}(\mathcal{E})$, and $T : C \to \mathbf{R}$. Then $TC = (C,T)$ is a *timed configuration* of $\mathcal{E}$ iff $\forall\ e \in C \diamond\ Eot(e) \leq T(e) \leq Lot(e)$ and $\forall\ e,e' \in C \diamond\ e < e' \ \Rightarrow\ T(e) \leq T(e')$. Informally speaking, the first condition expresses that an event can occur at a time when its timing constraints are met; and the second condition states that for any two events e and e' occurred if e causally precedes e', then e should temporally precede e'. We use $\mathbf{TC}(\mathcal{E})$ to denote the set of timed configurations of $\mathcal{E}$.

Let $\mathcal{E}$ be a timed event structure and $TC = (C,T), TC' = (C',T') \in \mathbf{TC}(\mathcal{E})$. We will write $TC \underset{d}{\overset{e}{\rightarrow}} TC'$ iff $C \cup \{e\} = C'$, and $T'|_C = T$ and $T'(e) = d$. A *run* of $\mathcal{E}$ is a sequence of the form $TC_0 \underset{d_1}{\overset{e_1}{\rightarrow}} TC_1 \underset{d_2}{\overset{e_2}{\rightarrow}} \ldots \underset{d_n}{\overset{e_n}{\rightarrow}} TC_n$, where $n \geq 0$ and $TC_0 = (\varnothing, \varnothing)$ is the initial timed configuration.

Now let us recall the notion of morphism between timed event structures.

Definition 16. *Let $\mathcal{E} = (E, <, Con, L, l, Eot, Lot)$ and $\mathcal{E}' = (E', <', Con', L', l', Eot', Lot')$ be timed event structures. A pair (μ, λ), where $\mu : E \to E'$ and $\lambda : L \to L'$ are functions, is called a* morphism, *if $l' \circ \mu = \lambda \circ l$ and for all $C \in \mathbf{C}(\mathcal{E})$ the following holds:*

- $\mu\ C \in \mathbf{C}(\mathcal{E}')$;
- $\forall e, e' \in C \diamond$ *if* $\mu(e) = \mu(e')$ *then* $e = e'$;
- $\forall e \in C \diamond Eot'(\mu(e)) \leq Eot(e)$ *and* $Lot(e) \leq Lot'(\mu(e))$.

Timed event structures and their morphisms form a category of timed event structures, **TES**.

Lemma 3. *Given a morphism $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ of* **TES**, *if $TC = (C,T)$ is a timed configuration of $\mathcal{E}$, then $TC' = (\mu\ C, T')$, where $T' \circ \mu = T$ will be a timed configuration of $\mathcal{E}'$.*

3.5. Timed Event Trees. The main goal of this paper is to expose an adjunction from the category of timed causal trees to the category of timed event structures. In order to achieve this aim, we will use a larger model, timed event trees, that embeds timed causal trees as well as timed event structures.

Definition 17. *A* timed event tree *$\mathcal{ET}$ is a tuple $(S, s_{in}, E, T, <, L, l, Eot, Lot)$, where (S, s_{in}, E, T) is a timed synchronization tree, $< \subseteq E \times E$ is a strict order, L is a set of labels, $l : E \to L$ is a labelling function, and $Eot,\ Lot : E \to \mathbf{R}$ are functions of the earliest and latest occurrence times of events, satisfying the following:*

(i) *for all $e \in E$ there exists a transition $(s, e, eot, lot, s') \in T$;*

(ii) *if $s \xrightarrow[eot,\ lot]{e} s'$ and $s \xrightarrow[eot',\ lot']{e} s''$, then $eot = eot'$, $lot = lot'$ and $s' = s''$;*

(iii) *if $s \xrightarrow[eot,\ lot]{e} s'$ and $u \xrightarrow[eot',\ lot']{e} u'$, then there is no sequence $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ such that $(s' = s_0 \wedge u = s_k)$ or $(u' = s_0 \wedge s = s_k)$;*

(iv) *if $e < e'$ and $s \xrightarrow[eot,\ lot]{e'} s'$ then there is a sequence $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ such that $e_1 = e$ and $s = s_k$;*

(v) *$Eot(e) \leq Lot(e)$, for all $e \in E$;*

(vi) $Eot(e) \leq eot \leq lot \leq Lot(e)$, *for all* $(s, e, eot, lot, s_1) \in T$.

We say two events e, e' of a timed event tree $\mathcal{ET}$ are *consistent* (denoted $e\ Con_{\mathcal{ET}}\ e'$) iff $e = e'$ or there exists a sequence $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ such that $(e_1 = e$ and $e_k = e')$ or $(e_1 = e'$ and $e_k = e)$.

A *run* of $\mathcal{ET}$ is a sequence $\gamma = \langle s_0, \nu_0 \rangle \overset{e_1}{\to} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \overset{e_n}{\to} \langle s_n, \nu_n \rangle$ such that $\nu_0 \leq \nu_1 \leq \ldots \leq \nu_n$ and for all $1 \leq i \leq n$ there is a transition $s_{i-1} \xrightarrow[eot_i,\ lot_i]{e_i} s_i$ such that $eot_i \leq \nu_i \leq lot_i$. Here, $s_0 = s_{in}$ and ν_0 is defined to be 0.

Lemma 4. *Let* $\mathcal{ET} = (S, s_{in}, E, T, <, L, l, Eot, Lot)$ *is a timed event tree and* $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{n-1} \xrightarrow[eot_n,\ lot_n]{e_n} s_n$ *for some* $n \geq 1$. *Then* $e_n \downarrow\subseteq \{e_1, \ldots, e_n\}$.

Now let us define the category of timed event trees.

Definition 18. *Let* $\mathcal{ET} = (S, s_{in}, E, T, <, L, l, Eot, Lot)$ *and* $\mathcal{ET}' = (S', s'_{in}, E', T', <', L', l', Eot', Lot')$ *be timed event trees. A pair* (μ, λ), *where* $\mu : E \to E'$ *and* $\lambda : L \to L'$ *are functions, is called a* morphism, *iff*

(i) $\mu(e) \downarrow\subseteq \mu(e\downarrow)$;

(ii) $l' \circ \mu = \lambda \circ l$;

(iii) *if* $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$, *then* $s'_{in} \xrightarrow[eot'_1,\ lot'_1]{\mu(e_1)} s'_1 \ldots s'_{k-1} \xrightarrow[eot'_k,\ lot'_k]{\mu(e_k)} s'_k$ *for some* $s'_j \in S'$ *and* $eot'_j \leq eot_j$ *and* $lot_j \leq lot'_j$ $(1 \leq j \leq k)$;

(iv) $Eot'(\mu(e)) \leq Eot(e)$ *and* $Lot(e) \leq Lot'(\mu(e))$, *for all* $e \in E$.

Lemma 5. *Given timed event trees* $\mathcal{ET} = (S, s_{in}, E, T, <, L, l, Eot, Lot)$ *and* $\mathcal{ET}' = (S', s'_{in}, E', T', <', L', l', Eot', Lot')$, *a morphism* $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ *generates the unique function* $\sigma_\mu : S \to S'$ *such that* (σ_μ, μ) *is a morphism between* (S, s_{in}, E, T) *and* (S', s'_{in}, E', T'), *and preserves concurrency: for all* $e, e' \in E$ *if* $e\ Con_{\mathcal{ET}}\ e'$ *and* $\mu(e) <' \mu(e')$, *then* $e < e'$.

Timed event trees and morphisms between them form the *category of timed event trees*, **TET**.

Lemma 6. *Given a morphism* $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ *of* **TET**, *if we have a run* $\gamma = \langle s_0, \nu_0 \rangle \overset{e_1}{\to} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \overset{e_n}{\to} \langle s_n, \nu_n \rangle$ *of* $\mathcal{T}$ *then* $\gamma' = \langle \sigma_\mu(s_0), \nu_0 \rangle \overset{\mu(e_1)}{\to} \langle \sigma_\mu(s_1), \nu_1 \rangle \ldots \langle \sigma_\mu(s_{n-1}), \nu_{n-1} \rangle \overset{\mu(e_n)}{\to} \langle \sigma_\mu(s_n), \nu_n \rangle$ *will be the run of* $\mathcal{ET}'$.

4. Relations Between Timed Models for Concurrency. In this section we investigate how the category of timed causal trees relates to the other timed model categories. In particular, we show that there is a coreflection from timed synchronization trees to timed causal trees, a coreflection from timed synchronization trees to timed event structures, a coreflection from timed causal trees to timed event trees, and a reflection from timed event trees to timed event structures. Thus, we will get the adjunction from timed causal trees to timed event structures which arises as the composition of a coreflection from timed causal trees to timed event trees and a reflection from timed event trees to timed event structures.

4.1. A coreflection between the categories TCT and TST. First, we investigate a relation between the categories **TCT** and **TST**. Clearly, any timed causal tree is a timed synchronization tree. Hence, we have a functor **c2s** : **TCT** → **TST** that forgets about the causality information and keeps morphisms. Moreover, it is easy to see that **c2s** is a faithful functor.

On the other hand, every timed synchronization tree determines a timed causal tree, in which the causal dependency relation is given by the order of the transitions in the tree. Now we can define a functor **s2c** : **TST** → **TCT**.

Definition 19. *Let* $\mathcal{S} = (S,\ s_{in},\ L,\ T)$ *and* $\mathcal{S}' = (S',\ s'_{in},\ L',\ T')$ *be timed synchronization trees and* $(\mu,\lambda) : \mathcal{S} \to \mathcal{S}'$ *be a morphism of* **TST**. *Define* $\mathbf{s2c}(\mathcal{S}) = (S,\ s_{in},\ L,\ T, <^*)$, *where* $(s, a, eot, lot, s') <^* (u,\ b,\ eot',\ lot',\ u')$ *if and only if there exists a sequence of transitions* $s' \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ *for some* $k \geq 1$ *such that* $s_k = u$; *and define* $\mathbf{s2c}(\mu,\lambda) = (\mu,\lambda)$.

Proposition 1. *The mapping* **s2c** *is a fully faithful functor.*

Доказательство. First, we note that $\mathbf{s2c}(\mathcal{S})$ is a timed causal tree for all timed synchronization trees $\mathcal{S} = (S,\ s_{in},\ L,\ T)$.

Second, we should check that $\mathbf{s2c}(\mu,\lambda) = (\mu,\lambda)$ is a morphism of **TCT** for all morphisms $(\mu,\lambda) : \mathcal{S} \to \mathcal{S}'$ of **TST**. We only need to prove that μ preserves concurrency. Let (s, a, eot, lot, s'), $(u, b, eot', lot', u') \in T$, $(s,\ a,\ eot,\ lot,\ s')$ Con $(u,\ b,\ eot',\ lot',\ u')$ and $(\mu(s)$, $\lambda(a),\ eot',\ lot',\ \mu(s')) <'^* (\mu(u),\ \lambda(b),\ eot',\ lot',\ u')$. This implies the existence of a sequence $\mu(s') \xrightarrow[eot'_1,\ lot'_1]{e'_1} s'_1 \ldots s'_{k-1} \xrightarrow[eot'_k,\ lot'_k]{e'_k} s'_k = \mu(u)$ for some $k \geq 1$. Hence, $(s, a, eot, lot, s') \neq (u, b, eot', lot', u')$, by the item (ii) of Definition 12. Furthermore, since $(s,\ a,\ eot,\ lot,\ s')$ Con $(u,\ b,\ eot',\ lot',\ u')$, we may conclude that either $(s, a, eot, lot, s') <^* (u, b, eot', lot', u')$ or $(u, b, eot', lot', u') <^* (s, a, eot, lot, s')$. Assume $(u,\ b,\ eot',\ lot',\ u') <^* (s,\ a,\ eot,\ lot,\ s')$. This means that there exists a sequence $u' \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{l-1} \xrightarrow[eot_l,\ lot_l]{e_l} s_l = s$ for some $l \geq 1$. This implies that $\mu(u') \xrightarrow[eot''_1,\ lot''_1]{\lambda(e_1)} \mu(s_1) \ldots \mu(s_{l-1}) \xrightarrow[eot''_l,\ lot''_l]{\lambda(e_l)} \mu(s_l) = \mu(s)$. This contradicts the item (ii) of Definition 12. Thus, $(s, a, eot, lot, s') <^* (u, b, eot', lot', u')$.

Third, consider an identity morphism $(1_S, 1_L) : \mathcal{S} \to \mathcal{S}$ and a pair of morphisms (σ,λ) : $\mathcal{S} \to \mathcal{S}'$ and $(\sigma',\lambda') : \mathcal{S}' \to \mathcal{S}''$ from **TST**. It is obvious that $\mathbf{s2c}(1_S,\ 1_L) = (1_S,\ 1_L)$ and $\mathbf{s2c}((\sigma',\lambda') \circ (\sigma,\lambda)) = \mathbf{s2c}(\sigma' \circ \sigma, \lambda' \circ \lambda) = (\sigma' \circ \sigma, \lambda' \circ \lambda) = \mathbf{s2c}(\sigma',\lambda') \circ \mathbf{s2c}(\sigma,\lambda)$. Thus, **s2c** is indeed a functor.

Finally, we need to clarify that **s2c** is a fully faithful functor. Take arbitrary objects $\mathcal{S}$ and $\mathcal{S}'$ of **TST**. Define a function $F_{\mathcal{S},\mathcal{S}'} : \mathbf{TST}(\mathcal{S},\mathcal{S}') \to \mathbf{TCT}(\mathbf{s2c}(\mathcal{S}),\mathbf{s2c}(\mathcal{S}'))$ such that $F_{\mathcal{S},\mathcal{S}'}(\sigma,\lambda) = \mathbf{s2c}(\sigma,\lambda) = (\sigma,\lambda)$ for all morphisms $(\sigma,\lambda) : \mathcal{S} \to \mathcal{S}'$ of **TST**. Since **s2c** is a functor, $F_{\mathcal{S},\mathcal{S}'}$ is a function. Moreover, it is easy to check that $F_{\mathcal{S},\mathcal{S}'}$ is injective, because $F_{\mathcal{S},\mathcal{S}'}(\sigma,\lambda) = (\sigma,\lambda)$. Hence, **s2c** is a faithful functor. Next, take an arbitrary morphism $(\sigma,\lambda) : \mathbf{s2c}(\mathcal{S}) \to \mathbf{s2c}(\mathcal{S}')$ of **TCT**. Clearly, (σ,λ) is a morphism of **TST** from $\mathcal{S}$ to $\mathcal{S}'$ and $F_{\mathcal{S},\mathcal{S}'}(\sigma,\lambda) = (\sigma,\lambda)$. Thus, **s2c** is a full functor. □

Proposition 2. *Let* $\mathcal{S} = (S,\ s_{in},\ L,\ T)$ *be a timed synchronization tree. Then* $\mathbf{s2c}(\mathcal{S})$ *is a timed causal tree,* $(1_S,1_L) : \mathcal{S} \to \mathbf{c2s}(\mathbf{s2c}(\mathcal{S}))$ *is an isomorphism and the pair* $(\mathbf{s2c}(\mathcal{S}),(1_S,1_L))$ *is a reflection of* $\mathcal{S}$ *along* **c2s**.

Доказательство. It is clear that $\mathbf{c2s}(\mathbf{s2c}(\mathcal{S})) = \mathcal{S}$. Hence, $(1_S,1_L) : \mathcal{S} \to \mathbf{c2s}(\mathbf{s2c}(\mathcal{S})) = \mathcal{S}$ is a morphism of **TST**. Moreover, it is an isomorphism.

Now we should prove that $(\mathbf{s2c}(\mathcal{S}),(1_S,1_L))$ is a reflection of $\mathcal{S}$ along **c2s**, i.e. whenever $\mathcal{C}'$ is a timed causal tree and $(\sigma,\lambda) : \mathcal{S} \to \mathbf{c2s}(\mathcal{C}')$ is a morphism of **TST**, then there exists a unique morphism $(g,\lambda') : \mathbf{s2c}(\mathcal{S}) \to \mathcal{C}'$ such that $(\sigma,\lambda) = \mathbf{c2s}(g,\lambda') \circ (1_S,1_L)$. Since $\mathbf{c2s}(g,\lambda') = (g,\lambda')$, we may conclude that λ' must be equal to λ and g must match σ. Hence, we should only show that $(\sigma,\lambda) : \mathbf{s2c}(\mathcal{S}) \to \mathcal{C}'$ is a morphism of **TCT**. Since $(\sigma,\lambda) : \mathcal{S} \to \mathbf{c2s}(\mathcal{C}')$ is a morphism of **TST**, we only need to check that σ preserves concurrency. Take an arbitrary (s,a,eot,lot,s'), $(u,b,eot^*,lot^*,u') \in T$ such that $(s,a,eot,lot,s')\ Con\ (u,b,eot^*,lot^*,u')$ and $(\sigma(s),\lambda(a),eot',lot',\sigma(s')) <' (\sigma(u),\lambda(b),eot'^*,lot'^*,\sigma(u'))$. Since $\mathcal{C}'$ is a timed causal tree, we may conclude that there exists a sequence $\sigma(s') \xrightarrow[eot'_1,\ lot'_1]{e'_1} \bar{s}_1 \ldots \bar{s}_{k-1} \xrightarrow[eot'_k,\ lot'_k]{e'_k} \sigma(u)$ for some $k \geq 1$.

Since $(s,a,eot,lot,s')\ Con\ (u,b,eot^*,lot^*,u')$, we have three admissible cases: $(s,\ a,\ eot,\ lot,\ s') = (u,\ b,\ eot^*,\ lot^*,\ u')$, $(s,\ a,\ eot,\ lot,\ s') <^* (u,\ b,\ eot^*,\ lot^*,\ u')$ and $(u,\ b,\ eot^*,\ lot^*,\ u') <^* (s,\ a,\ eot,\ lot,\ s')$. If $(s,\ a,\ eot,\ lot,\ s') = (u,\ b,\ eot^*,\ lot^*,\ u')$ then $(\sigma(s),\ \lambda(a),\ eot',\ lot',\ \sigma(s')) = (\sigma(u),\ \lambda(b),\ eot'^*,\ lot'^*,\ \sigma(u'))$, that contradicts our conditions. If $(u,\ b,\ eot^*,\ lot^*,\ u') <^* (s,\ a,\ eot,\ lot,\ s')$, we have a sequence $u' \xrightarrow[eot_1,\ lot_1]{e_1} \tilde{s}_1 \ldots \tilde{s}_{m-1} \xrightarrow[eot_m,\ lot_m]{e_m} s$ for some $m \geq 1$. Hence, $\sigma(u') \xrightarrow[eot''_1,\ lot''_1]{\lambda(e_1)} \sigma(\tilde{s}_1) \ldots \sigma(\tilde{s}_{m-1}) \xrightarrow[eot''_m,\ lot''_m]{\lambda(e_m)} \sigma(s)$. This contradicts the item (ii) of Definition 12. Hence, $(s,a,eot,lot,s') <^* (u,b,eot^*,lot^*,u')$.

Thus we can conclude that $(\mathbf{s2c}(\mathcal{S}),(1_S,1_L))$ is a reflection of $\mathcal{S}$ along **c2s**. □

The above results enable us to exhibit an adjunction between the categories **TST** and **TCT**.

Theorem 1. *The functor* **c2s** *is right adjoint to* **s2c** *and this adjunction is a coreflection.*

Доказательство. The first assertion follows from Proposition 2 and from the fact that for all morphisms $(\sigma,\lambda) : \mathcal{C} = (S,\ s_{in},\ L,\ T,<) \to \mathcal{C}' = (S',\ s'_{in},\ L',\ T',<')$ it is true that $(1_{S'},1_{L'}) \circ$

$(\sigma, \lambda) = (\sigma, \lambda) = \mathbf{s2c}(\mathbf{c2s}(\sigma, \lambda)) = \mathbf{s2c}(\mathbf{c2s}(\sigma, \lambda)) \circ (1_S, 1_L)$. Moreover, it follows from Proposition 2 that the unit ψ associates each timed synchronization tree $\mathcal{S} = (S,\ s_{in},\ L,\ T)$ with the isomorphism $(1_S, 1_L) : \mathcal{S} \to \mathbf{c2s}(\mathbf{s2c}(\mathcal{S}))$. Hence, ψ is a natural isomorphism. □

Thus, **TST** embeds fully and faithfully into **TCT** and is equivalent to the full subcategory of **TCT** consisting of those timed causal trees $\mathcal{C}$ that are isomorphic to $\mathbf{s2c}(\mathbf{c2s}(\mathcal{C}))$.

4.2. A coreflection between the categories TCT and TET. In this subsection we establish that there is a coreflection from timed causal trees to timed event trees. Note that any timed event tree gives rise to a timed causal tree by forgetting about events. Hence, we can specify a functor $\mathbf{et2c} : \mathbf{TET} \to \mathbf{TCT}$.

Definition 20. *Let $\mathcal{ET} = (S, s_{in}, E, T, <, L, l, Eot, Lot)$ and $\mathcal{ET}' = (S',\ s'_{in},\ E',\ T',\ <',\ L',\ l',\ Eot',\ Lot')$ be timed event trees and $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ be a morphism of* **TET**. *Define $\mathbf{et2c}(\mathcal{ET}) = (S, s_{in}, L, T^*, <^*)$, where $T^* = \{(s, l(e), eot, lot, s') \mid (s, e, eot, lot, s') \in T\}$, $(s, l(e), eot, lot, s') <^* (u,\ l(e'),\ eot',\ lot',\ u')$ if and only if $e < e'$ and there exists a sequence $s' \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} u$ for some $k \geq 0$; and define $\mathbf{et2c}(\mu, \lambda) = (\sigma_\mu, \lambda)$, where $\sigma_\mu : S \to S'$ is defined by μ as in Lemma 5.*

Proposition 3. *The mapping* **et2c** *is a faithful functor.*

Доказательство. It is clear that $\mathbf{et2c}(\mathcal{ET})$ is indeed a timed causal tree for all timed event trees $\mathcal{ET}$. The fact that $\mathbf{et2c}(\mu, \lambda) = (\sigma_\mu, \lambda)$ is a morphism of **TCT** for all morphisms $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ of **TET** follows from Lemma 5 and the equation $\lambda \circ l = l' \circ \mu$. Next, we consider an identity morphism $(1_E, 1_L) : \mathcal{ET} \to \mathcal{ET}$ and a pair of morphisms $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ and $(\mu', \lambda') : \mathcal{ET}' \to \mathcal{ET}''$ from **TET**. Obviously, $\mathbf{et2c}(1_E, 1_L) = (\sigma_{1_E}, 1_L) = (1_S, 1_L)$, where $(1_S, 1_L) : \mathbf{et2c}\mathcal{ET} \to \mathbf{et2c}\mathcal{ET}$ is an identity morphism of **TCT**, and $\mathbf{et2c}((\mu', \lambda') \circ (\mu, \lambda)) = \mathbf{et2c}(\mu' \circ \mu, \lambda' \circ \lambda) = (\sigma_{\mu' \circ \mu}, \lambda' \circ \lambda) = (\sigma_{\mu'} \circ \sigma_\mu, \lambda' \circ \lambda) = \mathbf{et2c}(\mu', \lambda') \circ \mathbf{et2c}(\mu, \lambda)$. Hence, we can conclude that **et2c** is a functor.

Now we need to show that the functor **et2c** is faithful. Take an arbitrary pair of objects $\mathcal{ET}$ and $\mathcal{ET}'$ of **TET**. Define a function $F_{\mathcal{ET},\mathcal{ET}'} : \mathbf{TET}(\mathcal{ET}, \mathcal{ET}') \to \mathbf{TCT}(\mathbf{et2c}(\mathcal{ET}), \mathbf{et2c}(\mathcal{ET}'))$ such that $F_{\mathcal{ET},\mathcal{ET}'}(\mu, \lambda) = \mathbf{et2c}(\mu, \lambda) = (\sigma_\mu, \lambda)$ for all morphisms $(\mu, \lambda) : \mathcal{ET} \to \mathcal{ET}'$ of **TET**. Clearly, $F_{\mathcal{ET},\mathcal{ET}'}$ is indeed a function, because **et2c** is a functor. Check that $F_{\mathcal{ET},\mathcal{ET}'}$ is injective. Take arbitrary two morphisms $(\mu_1, \lambda_1) : \mathcal{ET} \to \mathcal{ET}'$ and $(\mu_2, \lambda_2) : \mathcal{ET} \to \mathcal{ET}'$ such that $F_{\mathcal{ET},\mathcal{ET}'}(\mu_1, \lambda_1) = F_{\mathcal{ET},\mathcal{ET}'}(\mu_2, \lambda_2)$. This implies that $(\sigma_{\mu_1}, \lambda_1) = (\sigma_{\mu_2}, \lambda_2)$. Hence, $\lambda_1 = \lambda_2$ and $\sigma_{\mu_1} = \sigma_{\mu_2}$. Since σ_{μ_1} defines the function μ_1 in a unique way, we may conclude that $\mu_1 = \mu_2$. Hence, $F_{\mathcal{ET},\mathcal{ET}'}$ is injective, i.e. **et2c** is a faithful functor. □

Note, every timed causal tree $\mathcal{C}$ determines a timed event tree which is induced by $\mathcal{C}$ when we assume that each transition of $\mathcal{C}$ represents a separate event. This means that we take the transitions of $\mathcal{C}$ as events , and label each arc of $\mathcal{C}$ by the corresponding transition. This operation can be easily extended to a functor $\mathbf{c2et} : \mathbf{TCT} \to \mathbf{TET}$.

Definition 21. *Let* $\mathcal{C} = (S,\ s_{in},\ L,\ T, <)$ *and* $\mathcal{C}' = (S',\ s'_{in},\ L',\ T', <')$ *be timed causal trees and* $(\sigma,\lambda) : \mathcal{C} \to \mathcal{C}'$ *be a morphism of* $\mathbf{TCT}$*. Define* $\mathbf{c2et}(\mathcal{C}) = (S, s_{in}, T, T^\star, <, L, l, Eot, Lot)$*, where* $T^\star = \{(s,(s,a,eot,lot,s'),eot,lot,s') \mid (s,\ a,\ eot,\ lot,\ s') \in T\}$*,* $l(s,a,eot,lot,s') = a$*,* $Eot(s,a,eot,lot,s') = eot$ *and* $Lot(s,a,eot,lot,s') = lot$*; and define* $\mathbf{c2et}(\sigma,\lambda) = (\mu,\lambda)$*, where* $\mu : T \to T'$ *is given by the following equality:* $\mu(s,a,eot,lot,s') = (\sigma(s),\lambda(a),eot',lot',\sigma(s')) \in T'$ *for some* $eot' \le eot$ *and* $lot \le lot'$*.*

Proposition 4. *The mapping* $\mathbf{c2et}$ *is a faithful functor.*

Доказательство. It is easy to check that $\mathbf{c2et}(\mathcal{C}) = (S, s_{in}, T, T^\star, <, L, l, Eot, Lot)$ is a timed event tree for all timed causal trees $\mathcal{C} = (S,\ s_{in},\ L,\ T, <)$.

Now, we need to prove that $\mathbf{c2et}(\sigma,\lambda) : \mathbf{c2et}(\mathcal{C}) \to \mathbf{c2et}(\mathcal{C}')$ is a morphism of $\mathbf{TET}$ for all morphisms $(\sigma,\lambda) : \mathcal{C} \to \mathcal{C}'$ of $\mathbf{TCT}$. W.l.o.g. assume that $\mathcal{C} = (S,\ s_{in},\ L,\ T, <)$ and $\mathcal{C}' = (S',\ s'_{in},\ L',\ T', <')$. Then, $\mathbf{c2et}(\mathcal{C}) = (S,\ s_{in},\ T,\ T^\star,\ <,\ L,\ l,\ Eot,\ Lot)$ and $\mathbf{c2et}(\mathcal{C}')$ $= (S',\ s'_{in},\ T',\ T'^\star,\ <',\ L',\ l',\ Eot',\ Lot')$, where $T^\star = \{(s,\ (s,a,eot,lot,s'),\ eot,\ lot,\ s') \mid (s,a,eot,lot,s') \in T\}$, $l(s,a,eot,lot,s') = a$, $Eot(s,a,eot,lot,s') = eot$, $Lot(s,a,eot,lot,s') = lot$, $T'^\star = \{(u',\ (u',a',eot',lot',u''),\ eot',\ lot',\ u'') \mid (u',a',eot',lot',u'') \in T'\}$, $l'(u',a',eot',lot',u'') = a'$, $Eot'(u',a',eot',lot',u'') = eot'$ and $Lot'(u',\ a',\ eot',\ lot',\ u'') = lot'$. Moreover, $\mathbf{c2et}(\sigma,\lambda) = (\mu,\lambda)$, where μ associates $(s,\ a,\ eot,\ lot,\ s') \in T$ with some transition $(\sigma(s),\lambda(a),eot',lot',\sigma(s'))$ of $\mathbf{c2et}(\mathcal{C}')$ with $eot' \le eot$ and $lot \le lot'$. The existence and unicity of such transition follows from the item (ii) of Definition 11 and the item (iii) of Definition 12. Hence, $\mu : T \to T'$ and $\lambda : L \to L'$ are functions. Check that (μ,λ) satisfies the requirements of Definition 18.

(i) Let us show that $\mu(s,a,eot,lot,s') \downarrow \subseteq \mu((s,a,eot,lot,s') \downarrow)$.

 Take an arbitrary $(s,a,eot,lot,s') \in T$. Using the items (i), (iii) of Definition 12, we can find a unique sequence $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = s$ for some $k \ge 0$. Since (σ,λ) is a morphism of $\mathbf{TCT}$, we have $\sigma(s_{in}) = s'_{in} \xrightarrow[eot'_1,\ lot'_1]{\lambda(a_1)} \sigma(s_1) \ldots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{\lambda(a_k)} \sigma(s_k) = \sigma(s)$ $\xrightarrow[eot',\ lot']{\lambda(a)} \sigma(s')$ for some $eot'_1,\ \ldots,\ eot'_k,\ eot',\ lot'_1,\ \ldots,\ lot'_k,\ lot' \in \mathbf{R}$ such that $eot' \le eot$, $lot \le lot'$ and $eot'_j \le eot_j$ and $lot_j \le lot'_j$ for all $1 \le j \le k$. Clearly, $\mu(s,a,eot,lot,s') = (\sigma(s),$ $\lambda(a),\ eot',\ lot',\ \sigma(s'))$. Since $\mathbf{c2et}(\mathcal{C}')$ is a timed event tree, we have $\mu(s,a,eot,lot,s') \downarrow$ $\subseteq \{(\sigma(s_{in}) = s'_{in}, \lambda(a_1), eot'_1, lot'_1, \sigma(s_1)),\ \ldots,\ (\sigma(s_{k-1}), \lambda(a_k), eot'_k, lot'_k, \sigma(s))\}$ by Lemma 4. Hence, if $e' <' \mu(s,\ a,\ eot,\ lot,\ s')$ then $e' = (\sigma(s_{j-1}),\ \lambda(a_j),\ eot'_j,\ lot'_j,\ \sigma(s_j)) = \mu(s_{j-1},$ $a_j,\ eot_j,\ lot_j,\ s_j)$ for some $1 \le j \le k$. This implies $\mu(s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j) <' \mu(s,\ a,$ $eot,\ lot,\ s')$. According to Definition 14, it is easy to see that $(s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j)$

$< (s, a, eot, lot, s')$. Thus, $(s_{j-1}, a_j, eot_j, lot_j, s_j) \in (s, a, eot, lot, s') \downarrow$. Furthermore, $\mu(s, a, eot, lot, s') \downarrow \subseteq \mu((s, a, eot, lot, s') \downarrow)$.

(ii) It is clear that $l' \circ \mu(s, a, eot, lot, s') = l'(\sigma(s), \lambda(a), eot', lot', \sigma(s')) = \lambda(a) = \lambda \circ l(s, a, eot, lot, s')$ for all $(s, a, eot, lot, s') \in T$.

(iii) Let $s_{in} \underset{eot_1,\ lot_1}{\overset{(s_{in},a_1,eot_1,lot_1,s_1)}{\longrightarrow}} s_1 \ldots s_{k-1} \underset{eot_k,\ lot_k}{\overset{(s_{k-1},a_k,eot_k,lot_k,s_k)}{\longrightarrow}} s_k$ $(k \geq 0)$ in $\mathbf{c2et}(\mathcal{C})$. This means that $s_{in} \underset{eot_1,\ lot_1}{\overset{a_1}{\longrightarrow}} s_1 \ldots s_{k-1} \underset{eot_k,\ lot_k}{\overset{a_k}{\longrightarrow}} s_k$ in $\mathcal{C}$. Since (σ, λ) is a morphism of $\mathbf{TCT}$, we may conclude that $\sigma(s_{in}) = s'_{in} \underset{eot'_1,\ lot'_1}{\overset{\lambda(a_1)}{\longrightarrow}} \sigma(s_1) \ldots \sigma(s_{k-1}) \underset{eot'_k,\ lot'_k}{\overset{\lambda(a_k)}{\longrightarrow}} \sigma(s_k)$ in $\mathcal{C}'$ for some $eot'_1, \ldots, eot'_k, lot'_1, \ldots, lot'_k \in \mathbf{R}$ such that $eot'_j \leq eot_j$ and $lot_j \leq lot'_j$ for all $1 \leq j \leq k$. Hence, $\sigma(s_{in}) = s'_{in} \underset{eot'_1,\ lot'_1}{\overset{(\sigma(s_{in}),\lambda(a_1),eot'_1,lot'_1,\sigma(s_1))}{\longrightarrow}} \sigma(s_1) \ldots \sigma(s_{k-1}) \underset{eot'_k,\ lot'_k}{\overset{(\sigma(s_{k-1}),\lambda(a_k),eot'_k,lot'_k,\sigma(s_k))}{\longrightarrow}} \sigma(s_k)$ in $\mathbf{c2et}(\mathcal{C}')$ and for all $1 \leq j \leq k$ it holds that $\mu(s_{j-1}, a_j, eot_j, lot_j, s_j) = (\sigma(s_{j-1}), \lambda(a_j), eot'_j, lot'_j, \sigma(s_j))$.

(iv) Obviously, $Eot'(\mu(s, a, eot, lot, s')) = eot' \leq eot = Eot(s, a, eot, lot, s')$ and $Lot(s, a, eot, lot, s') = lot \leq lot' = Lot'(\mu(s, a, eot, lot, s'))$ for all $(s, a, eot, lot, s') \in T$.

This means that (μ, λ) is indeed a morphism of $\mathbf{TET}$ from $\mathbf{c2et}(\mathcal{C})$ to $\mathbf{c2et}(\mathcal{C}')$.

Next, we consider an identity morphism $(1_S, 1_L) : \mathcal{C} \to \mathcal{C}$ and a pair of morphisms $(\sigma, \lambda) : \mathcal{C} \to \mathcal{C}'$ and $(\sigma', \lambda') : \mathcal{C}' \to \mathcal{C}''$ from $\mathbf{TCT}$. Clearly, $\mathbf{c2et}(1_S, 1_L) = (\mu_{1_S, 1_L}, 1_L) = (1_T, 1_L)$, where $(1_T, 1_L) : \mathbf{c2et}\mathcal{C} \to \mathbf{c2et}\mathcal{C}$ is an identity morphism of $\mathbf{TET}$, and $\mathbf{c2et}((\sigma', \lambda') \circ (\sigma, \lambda)) = \mathbf{c2et}(\sigma' \circ \sigma, \lambda' \circ \lambda) = (\mu_{\sigma' \circ \sigma, \lambda' \circ \lambda}, \lambda' \circ \lambda) = (\mu_{\sigma', \lambda'} \circ \mu_{\sigma, \lambda}, \lambda' \circ \lambda) = \mathbf{c2et}(\sigma', \lambda') \circ \mathbf{c2et}(\sigma, \lambda)$. Thus, $\mathbf{c2et}$ is indeed a functor.

In conclusion we prove that the functor $\mathbf{c2et}$ is faithful. Take an arbitrary pair of timed causal trees $\mathcal{C}$ and $\mathcal{C}'$. Define a function $F_{\mathcal{C},\mathcal{C}'} : \mathbf{TCT}(\mathcal{C}, \mathcal{C}') \to \mathbf{TET}(\mathbf{c2et}(\mathcal{C}), \mathbf{c2et}(\mathcal{C}'))$ such that $F_{\mathcal{C},\mathcal{C}'}(\sigma, \lambda) = \mathbf{c2et}(\sigma, \lambda) = (\mu_{\sigma,\lambda}, \lambda)$ for all morphisms $(\sigma, \lambda) : \mathcal{C} \to \mathcal{C}'$ of $\mathbf{TCT}$. It is easy to see that $F_{\mathcal{C},\mathcal{C}'}$ is indeed a function, because $\mathbf{c2et}$ is a functor. Verify that $F_{\mathcal{C},\mathcal{C}'}$ is an injective function. Take arbitrary two morphisms $(\sigma_1, \lambda_1) : \mathcal{C} \to \mathcal{C}'$ and $(\sigma_2, \lambda_2) : \mathcal{C} \to \mathcal{C}'$ such that $F_{\mathcal{C},\mathcal{C}'}(\sigma_1, \lambda_1) = F_{\mathcal{C},\mathcal{C}'}(\sigma_2, \lambda_2)$. This implies $(\mu_{\sigma_1,\lambda_1}, \lambda_1) = (\mu_{\sigma_2,\lambda_2}, \lambda_2)$. Hence, $\lambda_1 = \lambda_2$ and $\mu_{\sigma_1,\lambda_1} = \mu_{\sigma_2,\lambda_2}$. Contemplate an arbitrary state $s \in S$. Since $\mathcal{C}$ is a timed synchronization tree, we have some transition (s', a, eot, lot, s) of $\mathcal{C}$. Clearly, for all $i = 1, 2$ $\mu_{\sigma_i,\lambda_i}(s', a, eot, lot, s) = (\sigma_i(s'), \lambda_i(a), eot_i, lot_i, \sigma_i(s)) \in T'$. Since $\mu_{\sigma_1,\lambda_1} = \mu_{\sigma_2,\lambda_2}$, we have $(\sigma_1(s'), \lambda_1(a), eot_1, lot_1, \sigma_1(s))$ $= (\sigma_2(s'), \lambda_2(a), eot_2, lot_2, \sigma_2(s))$. Hence, $\sigma_1(s) = \sigma_2(s)$. This fact implies $\sigma_1 = \sigma_2$. Thus, $F_{\mathcal{C},\mathcal{C}'}$ is injective, i.e. $\mathbf{c2et}$ is a faithful functor. □

Proposition 5. *Let* $\mathcal{C} = (S, s_{in}, L, T, <)$ *be a timed causal tree. Then* $\mathbf{c2et}(\mathcal{C})$ *is a timed event tree,* $(1_S, 1_L) : \mathcal{C} \to \mathbf{et2c}(\mathbf{c2et}(\mathcal{C}))$ *is an isomorphism and the pair* $(\mathbf{c2et}(\mathcal{C}), (1_S, 1_L))$ *is a reflection of* $\mathcal{C}$ *along* $\mathbf{et2c}$.

Доказательство. Since **c2et** is a functor, $\mathbf{c2et}(\mathcal{C})$ is a timed event tree. Obviously, $\mathbf{c2et}(\mathcal{C}) = (S,\ s_{in},\ T,\ T^\star,\ <,\ L,\ l,\ Eot,\ Lot)$, where $T^\star = \{(s,\ (s,\ a,\ eot,\ lot,\ s'),\ eot,\ lot,\ s') \mid (s,\ a,\ eot,\ lot,\ s') \in T\}$, $l(s,a,eot,lot,s') = a$, $Eot(s,a,eot,lot,s') = eot$ and $Lot(s,a,eot,lot,s') = lot$. Contemplate a timed causal tree $\mathbf{et2c}(\mathbf{c2et}(\mathcal{C}))$. Clearly, $\mathbf{et2c}(\mathbf{c2et}(\mathcal{C})) = (S, s_{in}, L, T^{**}, <^{**})$, where $T^{**} = \{(s,\ l(e),\ eot,\ lot,\ s') \mid e \in T \text{ and } (s,\ e,\ eot,\ lot,\ s') \in T^\star\}$ and $(s,\ l(e),\ eot,\ lot,\ s') <^{**} (u,\ l(e'),\ eot',\ lot',\ u') \iff e, e' \in T,\ e < e'$ and $\exists\ s' \xrightarrow[eot_1,\ lot_1]{(s',a_1,eot_1,lot_1,s_1)} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{(s_{k-1},a_k,eot_k,lot_k,s_k)} s_k = u$ $(k \geq 0)$ in $\mathbf{c2et}(\mathcal{C})$. Hence, $T^{**} = \{(s, l(s,a,eot,lot,s'), eot, lot, s') \mid (s,a,eot,lot,s') \in T\} = T$. Moreover, it holds that $(s,\ l(e),\ eot,\ lot,\ s') <^{**} (u,\ l(e') = b,\ eot',\ lot',\ u') \iff e = (s,\ a,\ eot,\ lot,\ s'),\ e' = (u,\ b,\ eot',\ lot',\ u'),\ (s,\ a,\ eot,\ lot,\ s') < (u,\ b,\ eot',\ lot',\ u')$ and $\exists\ s' \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = u$ $(k \geq 0)$ in $\mathcal{C}$. Because $\mathcal{C}$ is a timed causal tree, we have $(s,\ l((s,a,eot,lot,s')) = a,\ eot,\ lot,\ s') <^{**} (u,\ l((u,b,eot',lot',u')) = b,\ eot',\ lot',\ u') \iff (s,\ a,\ eot,\ lot,\ s') < (u,\ b,\ eot',\ lot',\ u')$, i.e. $< = <^{**}$. Thus, $\mathbf{et2c}(\mathbf{c2et}(\mathcal{C})) = \mathcal{C}$.

Clearly, $(1_S, 1_L) : \mathcal{C} \to \mathbf{et2c}(\mathbf{c2et}(\mathcal{C})) = \mathcal{C}$ is a morphism of **TCT**. Furthermore, it is an isomorphism.

Now we should prove that $(\mathbf{c2et}(\mathcal{C}), (1_S, 1_L))$ is a reflection of $\mathcal{C}$ along **et2c**, i.e. whenever $\mathcal{ET}'$ is a timed event tree and $(\sigma, \lambda) : \mathcal{C} \to \mathbf{et2c}(\mathcal{ET}')$ is a morphism of **TCT**, there exists a unique morphism $(g, \lambda') : \mathbf{c2et}(\mathcal{C}) \to \mathcal{ET}'$ such that $(\sigma, \lambda) = \mathbf{et2c}(g, \lambda') \circ (1_S, 1_L)$. Since $\mathbf{et2c}(g, \lambda') = (\sigma_g, \lambda')$, we may conclude that λ' must be equal to λ and g must be defined so that $\sigma_g = \sigma$.

W.l.o.g. assume that $\mathcal{ET}' = (S', s'_{in}, E', T', <', L', l', Eot', Lot')$ and $(\sigma, \lambda) : \mathcal{C} \to \mathbf{et2c}(\mathcal{ET}')$ is a morphism of **TCT**. Obviously, $\mathbf{et2c}(\mathcal{ET}') = (S',\ s'_{in},\ L',\ T'^{*},\ <'^{*})$, where $T'^{*} = \{(u,\ l'(e),\ eot,\ lot,\ u') \mid (u,\ e,\ eot,\ lot,\ u') \in T'\}$ and $(u, l'(e), eot, lot, u') <'^{*} (t, l'(e'), eot', lot', t') \iff e <' e'$ and there exists a sequence $u' \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} t$ for some $k \geq 0$. Define a mapping $g : T \to E'$ as follows: $g(s, a, eot, lot, s') = e'$ such that $l'(e') = \lambda(a)$ and $(\sigma(s), e', eot', lot', \sigma(s')) \in T'$ for some $eot', lot' \in \mathbf{R}$ with $eot' \leq eot$ and $lot \leq lot'$.

First, check that g is a function. Let $(s, a, eot, lot, s') \in T$. Since (σ, λ) is a morphism of **TCT**, we have that $(\sigma(s),\ \lambda(a),\ eot',\ lot',\ \sigma(s')) \in T'^{*}$ for some $eot', lot' \in \mathbf{R}$ such that $eot' \leq eot$ and $lot \leq lot'$. This implies that $(\sigma(s),\ e',\ eot',\ lot',\ \sigma(s')) \in T'$ for some $e' \in E'$ such that $l'(e') = \lambda(a)$. Hence, for all $(s, a, eot, lot, s') \in T$ there is an event e' such that $l'(e') = \lambda(a)$ and $(\sigma(s), e', eot', lot', \sigma(s')) \in T'$ for some $eot', lot' \in \mathbf{R}$ with $eot' \leq eot$ and $lot \leq lot'$. Suppose that we have $e', e'' \in E'$ such that $(\sigma(s), e', eot', lot', \sigma(s'))$, $(\sigma(s), e'', eot'', lot'', \sigma(s')) \in T'$, $eot' \leq eot$, $lot \leq lot'$, $eot'' \leq eot$, $lot \leq lot''$ and $l'(e') = l'(e'') = \lambda(a)$. Due to the item (iii) of Definition 12, it holds that $e' = e''$. Thus, g is well defined.

Second, establish that $(g, \lambda) : \mathbf{c2et}(\mathcal{C}) \to \mathcal{ET}'$ is a morphism of **TET**.

- Check that $g(s, a, eot, lot, s') \downarrow \subseteq g((s, a, eot, lot, s') \downarrow)$.

Assume $e'' \in g(s, a, eot, lot, s') \downarrow$. It means that $e'' <' e' = g(s, a, eot, lot, s')$ with $(\sigma(s), e', eot', lot', \sigma(s')) \in T'$ and $l'(e') = \lambda(a)$. Due to the items (i), (iii) of Definition 12, we have a unique sequence $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = s$ in $\mathcal{C}$. It means that $s_{in} \xrightarrow[eot_1,\ lot_1]{(s_{in},a_1,eot_1,lot_1,s_1)} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{(s_{k-1},a_k,eot_k,lot_k,s_k)} s_k = s \xrightarrow[eot,\ lot]{(s,a,eot,lot,s')} s'$ in $\mathbf{c2et}(\mathcal{C})$. From Lemma 4 we get $(s,a,eot,lot,s') \downarrow \subseteq \{(s_{in},a_1,eot_1,lot_1,s_1),\dots,(s_{k-1},a_k,eot_k,lot_k,s_k = s),(s,a,eot,lot,s')\}$. Since (σ,λ) is a morphism, it holds that $\sigma(s_{in}) = s'_{in} \xrightarrow[eot'_1,\ lot'_1]{\lambda(a_1)} \sigma(s_1) \dots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{\lambda(a_k)} \sigma(s_k) = \sigma(s) \xrightarrow[eot',\ lot']{\lambda(a)} \sigma(s')$ in $\mathbf{et2c}(\mathcal{ET}')$ for some eot', eot'_1, …, eot'_k, lot', lot'_1, …, $lot'_k \in \mathbf{R}$ such that $eot' \le eot$, $lot \le lot'$, and $eot'_j \le eot_j$ and $lot_j \le lot'_j$ for all $1 \le j \le k$. Hence, $\sigma(s_{in}) = s'_{in} \xrightarrow[eot'_1,\ lot'_1]{e'_1} \sigma(s_1) \dots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{e'_k} \sigma(s_k) = \sigma(s) \xrightarrow[eot',\ lot']{e'} \sigma(s')$ in $\mathcal{ET}'$ for some e', $e'_1,\dots,e'_k \in E'$ such that $l'(e') = \lambda(a)$, $l'(e'_j) = \lambda(a_j)$ for all $1 \le j \le k$. Moreover, it is easy to see that $g(s,a,eot,lot,s') = e'$, $g(s_{j-1}, a_j, eot_j, lot_j, s_j) = e'_j$ $(1 \le j \le k)$. Since $\mathcal{ET}'$ is a timed event tree, we may conclude that $g(s,a,eot,lot,s') \downarrow \subseteq \{g(s_{in},a_1,eot_1,lot_1,s_1), \dots, g(s_{k-1}, a_k, eot_k, lot_k, s_k = s)\}$ by Lemma 4. Assume $e'' \in g(s,a,eot,lot,s') \downarrow$. It means that $e'' = g(s_{j-1},a_j,eot_j,lot_j,s_j) = e'_j$ for some $1 \le j \le k$. This implies that $(\sigma(s_{j-1}),e'_j,eot'_j,lot'_j,\sigma(s_j)) <'^* (\sigma(s),e',eot',lot',\sigma(s'))$. According to Definition 14, it holds that $(s_{j-1},e_j,eot_j,lot_j,s_j) < (s,e,eot,lot,s')$. Thus, $e'' \in g((s,a,eot,lot,s') \downarrow)$.

- Obviously, $l' \circ g(s,a,eot,lot,s') = l'(e') = \lambda(a) = \lambda \circ l(s,a,eot,lot,s')$.

- Let $s_{in} \xrightarrow[eot_1,\ lot_1]{(s_{in},a_1,eot_1,lot_1,s_1)} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{(s_{k-1},a_k,eot_k,lot_k,s_k)} s_k$ $(k \ge 0)$ in $\mathbf{c2et}(\mathcal{C})$. This means that $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k$ in $\mathcal{C}$. Since (σ,λ) is a morphism, we may conclude that $\sigma(s_{in}) = s'_{in} \xrightarrow[eot'_1,\ lot'_1]{\lambda(a_1)} \sigma(s_1) \dots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{\lambda(a_k)} \sigma(s_k)$ in $\mathbf{et2c}(\mathcal{ET}')$ for some eot'_1, …, eot'_k, lot'_1, …, $lot'_k \in \mathbf{R}$ such that $eot'_j \le eot_j$ and $lot_j \le lot'_j$ for all $1 \le j \le k$. This implies that $\sigma(s_{in}) = s'_{in} \xrightarrow[eot'_1,\ lot'_1]{e'_1} \sigma(s_1) \dots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{e'_k} \sigma(s_k)$ in $\mathcal{ET}'$ for some $e'_1,\dots,e'_k \in E'$ such that $l'(e'_j) = \lambda(a_j)$ for all $1 \le j \le k$. Moreover, it is easy to see that $g(s_{j-1}, a_j, eot_j, lot_j, s_j) = e'_j$ $(1 \le j \le k)$.

- Note that $Eot'(g(s,a,eot,lot,s')) \le eot' \le eot = Eot(s,a,eot,lot,s')$ and $Lot(s, a, eot, lot, s') = lot \le lot' \le Lot'(g(s,a,eot,lot,s'))$.

Thus, (g,λ) is indeed a morphism of $\mathbf{TET}$ from $\mathbf{c2et}(\mathcal{C})$ to $\mathcal{ET}'$.

It is easy to see that $(\sigma,\lambda) = (\sigma_g,\lambda)$ and g is a unique function such that $\sigma_g = \sigma$. Furthermore, $(\mathbf{c2et}(\mathcal{C}),(1_S,1_L))$ is a reflection of $\mathcal{C}$ along $\mathbf{et2c}$. □

Now we can summarize the obtained results in order to introduce an adjunction between **TET** and **TCT**.

Theorem 2. *The functor* **et2c** *is right adjoint to* **c2et** *and this adjunction is a coreflection.*

Доказательство. The first statement follows from Proposition 5 and from the fact that for all morphisms $(\sigma, \lambda) : \mathcal{C} = (S,\ s_{in},\ L,\ T, <) \to \mathcal{C}' = (S',\ s'_{in},\ L',\ T', <')$ it is true that $(1_{S'}, 1_{L'}) \circ (\sigma, \lambda) = (\sigma, \lambda) = \mathbf{et2c}(\mathbf{c2et}(\sigma, \lambda)) = \mathbf{et2c}(\mathbf{c2et}(\sigma, \lambda)) \circ (1_S, 1_L)$. Next, due to Proposition 5 we may conclude that the unit ψ associates each timed causal tree $\mathcal{C} = (S,\ s_{in},\ L,\ T, <)$ with the isomorphism $(1_S, 1_L) : \mathcal{C} \to \mathbf{et2c}(\mathbf{c2et}(\mathcal{C}))$. Hence, ψ is a natural isomorphism. □

Thus, **TCT** embeds fully and faithfully into **TET** and is equivalent to the full subcategory of **TET** consisting of those timed event trees $\mathcal{ET}$ that are isomorphic to $\mathbf{c2et}(\mathbf{et2c}(\mathcal{ET}))$.

4.3. A reflection between the categories TES and TET. This subsection is dedicated to investigation of the categories **TES** and **TET** and a relation between them. The runs of a timed event structure can be ordered in a tree. Hence, any timed event structure forms a timed event tree whose states are the runs of the timed event structure. This gives rise to a functor $\mathbf{e2et} : \mathbf{TES} \to \mathbf{TET}$.

Definition 22. *Let* $\mathcal{E} = (E,\ <,\ Con,\ L,\ l,\ Eot,\ Lot)$ *and* $\mathcal{E}' = (E',\ <',\ Con',\ L',\ l',\ Eot',\ Lot')$ *be timed event structures and* $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ *be a morphism from* **TES**. *Define* $\mathbf{e2et}(\mathcal{E}) = (S,\ \epsilon,\ E,\ Tran,\ <,\ L,\ l,\ Eot,\ Lot)$, *where* $S = \{e_1 \dots e_n \in E^* \mid n \geq 0, \{e_1, \dots, e_n\} \in \mathbf{C}(\mathcal{E})$ *and for all* $1 \leq i, j \leq n$ *if* $e_i < e_j$ *then* $i < j\}$ *and* $Tran = \{(e_1 \dots e_n, e_{n+1}, Eot(e_{n+1}), Lot(e_{n+1}), e_1 \dots e_n e_{n+1}) \mid e_1 \dots e_n,\ e_1 \dots e_n e_{n+1} \in S\}$; *and define* $\mathbf{e2et}(\mu, \lambda) = (\mu, \lambda)$.

Proposition 6. *The mapping* **e2et** *is a fully faithful functor.*

Доказательство. First, we need to show that $\mathbf{e2et}(\mathcal{E})$ is a timed event tree for all timed event structures $\mathcal{E}$. Using the definition of the sets S and $Tran$, we may easy check that $(S,\ \epsilon,\ E,\ Tran)$ is a timed synchronization tree. Note that $< \subseteq E \times E$ is a strict order, because $\mathcal{E}$ is a timed event structure. Next, we should prove that $\mathbf{e2et}(\mathcal{E})$ satisfies the requirements of Definition 17:

(i) for all $e \in E$ there exists a transition $(s, e, eot, lot, s') \in Tran$.

Clearly, $C = e \downarrow \cup \{e\} \in \mathbf{C}(\mathcal{E})$. W.l.o.g. assume that $C = \{e_1, \dots, e_n\}$ for some $n \geq 0$ such that $e_n = e$ and for all $1 \leq i, j \leq n$ if $(e_i < e_j)$ then $i < j$. Define $s_i = e_1 \dots e_i$ for all $1 \leq i \leq n$ and $s_0 = s_{in} = \epsilon$. Obviously, for all $1 \leq i \leq n$, $s_i \in S$ and $(s_{i-1}, e_i, Eot(e_i), Lot(e_i), s_i) \in Tran$.

(ii) if $(s, e, eot, lot, s'), (s, e, eot', lot', s'') \in Tran$, then $(s,\ e,\ eot,\ lot,\ s') = (s,\ e,\ eot',\ lot',\ s'')$.

Due to the definition of the set $Tran$, we have that $s = e_1^* \dots e_m^*$ for some $m \geq 0$, $s' = e_1^* \dots e_m^* e$, $s'' = e_1^* \dots e_m^* e$ and $eot = eot' = Eot(e)$ and $lot = lot' = Lot(e)$. Hence, $(s, e, eot, lot, s') = (s, e, eot', lot', s'')$.

(iii) if (s, e, eot, lot, s'), $(u, e, eot', lot', u') \in Tran$, then there is no sequence $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ such that $(s' = s_0 \wedge u = s_k)$ or $(u' = s_0 \wedge s = s_k)$.

Suppose that (s, e, eot, lot, s'), $(u, e, eot', lot', u') \in Tran$. By the construction of the set $Tran$, we have that $s = e_1^* \ldots e_m^*$ for some $m \geq 0$, $s' = e_1^* \ldots e_m^* e$, $u = e_1' \ldots e_k'$ for some $k \geq 0$ and $u' = e_1' \ldots e_k' e$. This means that $e \in s'$, $e \in u'$, $e \notin s$ and $e \notin u$. It is easy to see that if $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ then s_0 is a prefix of s_k. Hence, if $s' = s_0$ then $u \neq s_k$ and if $u' = s_0$ then $s \neq s_k$.

(iv) if $e < e'$ and $(s, e', eot, lot, s') \in Tran$, then there is a sequence $s_0 \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ such that $e_1 = e$ and $s = s_k$.

Since $(s, e', eot, lot, s') \in Tran$ we have that $s = e_1^* \ldots e_m^*$ for some $m \geq 0$, $s' = e_1^* \ldots e_m^* e'$ and $\{e_1^*, \ldots, e_m^*\}$, $\{e_1^*, \ldots, e_m^*, e'\} \in \mathbf{C}(\mathcal{E})$. Hence, $e \in \{e_1^*, \ldots, e_m^*\}$. W.l.o.g. assume $e = e_j^*$ for some $1 \leq j \leq m$. Define $s_i^* = e_1^* \ldots e_i^*$ for all $1 \leq i \leq m$ and $s_{m+1}^* = e_1^* \ldots e_m^* e'$. Clearly, $s_{j-1}^* \xrightarrow[Eot(e_j),\ Lot(e_j)]{e_j = e} s_j^* \ldots s_{m-1}^* \xrightarrow[Eot(e_m),\ Lot(e_m)]{e_m} s_m^* \xrightarrow[Eot(e'),\ Lot(e')]{e'} s_{m+1}^* (m \geq 0)$.

(v) $Eot(e) \leq Lot(e)$ for all $e \in E$.

This follows from the fact that $\mathcal{E}$ is a timed event structure.

(vi) for all $(s, e, eot, lot, s_1) \in Tran$ $Eot(e) \leq eot \leq lot \leq Lot(e)$.

Clearly, for all $(s, e, eot, lot, s_1) \in Tran$ $Eot(e) = eot \leq lot = Lot(e)$.

Thus, $\mathbf{e2et}(\mathcal{E})$ is a timed event tree.

Second, we need to prove that $\mathbf{e2et}(\mu, \lambda) : \mathbf{e2et}(\mathcal{E}) \to \mathbf{e2et}(\mathcal{E}')$ is a morphism of **TET** for all morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ of **TES**. W.l.o.g. assume that $\mathcal{E} = (E, <, Con, L, l, Eot, Lot)$ and $\mathcal{E}' = (E', <', Con', L', l', Eot', Lot')$. Then, $\mathbf{e2et}(\mathcal{E}) = (S, \epsilon, E, Tran, <, L, l, Eot, Lot)$ and $\mathbf{e2et}(\mathcal{E}') = (S', \epsilon, E', Tran', <', L', l', Eot', Lot')$. Since $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ is a morphism of **TES**, we get $\mu : E \to E'$ and $\lambda : L \to L'$ are functions and $l' \circ \mu = \lambda \circ l$. Check that (μ, λ) satisfies the requirements of Definition 18.

- Let us show that $\mu(e) \downarrow \subseteq \mu(e \downarrow)$.

  Take an arbitrary $e \in E$. Obviously, $C = e \downarrow \cup \{e\} \in \mathbf{C}(\mathcal{E})$. Hence, $\mu\ C \in \mathbf{C}(\mathcal{E}')$. Since $\mu(e) \in \mu\ C$, we have that $\mu(e) \downarrow \subseteq \mu\ C = \mu(e \downarrow) \cup \{\mu(e)\}$. Because (μ, λ) is a morphism of **TES**, we have $\mu(e) \downarrow \cap \{\mu(e)\} = \varnothing$. Thus, $\mu(e) \downarrow \subseteq \mu(e \downarrow)$.

- Let $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{n-1} \xrightarrow[eot_n,\ lot_n]{e_n} s_n$ for some $n \geq 0$. Due to the definition of the set $Tran$, we have that $s_{in} = \epsilon$, and for all $1 \leq i \leq n$ $s_i = e_1 \ldots e_i \in S$, $eot_i = Eot(e_i)$ and $lot_i = Lot(e_i)$. Since $s_i \in S$ $(1 \leq i \leq n)$, we get $\{e_1, \ldots, e_i\} \in \mathbf{C}(\mathcal{E})$ for all $1 \leq i \leq n$. Because (μ, λ) is a morphism of **TES**, it holds that $\{\mu(e_1), \ldots, \mu(e_i)\} \in \mathbf{C}(\mathcal{E}')$ for all $1 \leq i \leq k$. Define

$s'_i = \mu(e_1)\dots\mu(e_i)$ for all $1 \le i \le n$. Clearly, $s'_i \in S'$ $(1 \le i \le k)$ and $s'_{in} \xrightarrow[Eot'(\mu(e_1)),\ Lot'(\mu(e_1))]{\mu(e_1)} s'_1 \dots s'_{n-1} \xrightarrow[Eot'(\mu(e_n)),\ Lot'(\mu(e_n))]{\mu(e_n)} s'_n$, $Eot'(\mu(e_j)) \le Eot(e_j)$ and $Lot(e_j) \le Lot'(\mu(e_j))$ for all $1 \le j \le k$.

- Clearly, $Eot'(\mu(e)) \le Eot(e)$ and $Lot(e) \le Lot'(\mu(e))$ for all $e \in E$, since (μ, λ) is a morphism of **TES**.

This means that (μ, λ) is indeed a morphism of **TET** from $\mathbf{e2et}(\mathcal{E})$ to $\mathbf{e2et}(\mathcal{E}')$.

Third, consider an identity morphism $(1_E, 1_L) : \mathcal{E} \to \mathcal{E}$ and a pair of morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ and $(\mu', \lambda') : \mathcal{E}' \to \mathcal{E}''$ from **TES**. Clearly, $\mathbf{e2et}(1_E, 1_L) = (1_E, 1_L)$, and $\mathbf{e2et}((\mu', \lambda') \circ (\mu, \lambda))$ $= \mathbf{e2et}(\mu' \circ \mu, \lambda' \circ \lambda) = (\mu' \circ \mu, \lambda' \circ \lambda) = (\mu', \lambda') \circ (\mu, \lambda) = \mathbf{e2et}((\mu', \lambda')) \circ \mathbf{e2et}(\mu, \lambda)$. Thus, we have that **e2et** is indeed a functor.

Finally, we need to show that **e2et** is a fully faithful functor. Take an arbitrary objects $\mathcal{E}$ and $\mathcal{E}'$ of **TES**. Define a function $F_{\mathcal{E},\mathcal{E}'} : \mathbf{TES}(\mathcal{E}, \mathcal{E}') \to \mathbf{TET}(\mathbf{e2et}(\mathcal{E}), \mathbf{e2et}(\mathcal{E}'))$ such that $F_{\mathcal{E},\mathcal{E}'}(\mu, \lambda) = \mathbf{e2et}(\mu, \lambda) = (\mu, \lambda)$ for all morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ of **TES**. It is obvious that $F_{\mathcal{E},\mathcal{E}'}$ is a function, because **e2et** is a functor. It is easy to see that $F_{\mathcal{E},\mathcal{E}'}$ is injective, because $F_{\mathcal{E},\mathcal{E}'}(\mu, \lambda) = (\mu, \lambda)$. Hence, **e2et** is a faithful functor. Check that $F_{\mathcal{E},\mathcal{E}'}$ is a surjective function. Take an arbitrary morphism $(\mu, \lambda) : \mathbf{e2et}(\mathcal{E}) \to \mathbf{e2et}(\mathcal{E}')$ of **TET**. Since (μ, λ) is a morphism of **TET**, we may conclude that $\mu : E \to E'$ and $\lambda : L \to L'$ are functions, $l' \circ \mu = \lambda \circ l$ and $Eot'(\mu(e)) \le Eot(e)$ and $Lot(e) \le Lot'(\mu(e))$ for all $e \in E$. Let C be a configuration of $\mathcal{E}$. By the definition of the sets of states of $\mathbf{e2et}(\mathcal{E})$ and $\mathbf{e2et}(\mathcal{E}')$, we get that $\mu\ C \in \mathbf{C}(\mathcal{E}')$ and for all $e, e' \in C$ if $\mu(e) = \mu(e')$ then $e = e'$. This implies that (μ, λ) is a morphism of **TES** and $F_{\mathcal{E},\mathcal{E}'}(\mu, \lambda) = (\mu, \lambda)$. Therefore, **e2et** is a full functor. □

Note that we can transform any timed event tree into a timed event structure, defining the set of consistent events as a set of events that appear together on some branch and ignoring the tree structure. Thus we obtain a functor $\mathbf{et2e} : \mathbf{TET} \to \mathbf{TES}$.

Definition 23. *Let* $\mathcal{ET} = (S,\ s_{in},\ E,\ T,\ <,\ L,\ l,\ Eot,\ Lot)$ *and* $\mathcal{ET}' = (S',\ s'_{in},\ E',\ T',\ <',\ L',\ l',\ Eot',\ Lot')$ *be timed event trees. Define* $\mathbf{et2e}(\mathcal{ET})$ *as* $(E,\ <,\ Con,\ L,\ l,\ Eot,\ Lot)$*, where* Con *exactly contains all subsets* A *of the sets* $\{e_1, \dots, e_k\} \subseteq E$ $(k \ge 0)$ *such that there are states* $s_1,\ \dots,\ s_k \in S$ *with* $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ *for some* $eot_1, \dots, eot_k,\ lot_1, \dots, lot_k \in \mathbf{R}$. *Moreover,* $\mathbf{et2e}(\mu, \lambda) = (\mu, \lambda)$.

Proposition 7. *The mapping* **et2e** *is a faithful functor.*

Доказательство. First, we need to show that $\mathbf{et2e}(\mathcal{ET})$ is a timed event structure for all timed event trees $\mathcal{ET}$. It follows from Definition 17 and Lemma 4.

Second, we have to prove that $\mathbf{et2e}(\mu,\lambda) : \mathbf{et2e}(\mathcal{ET}) \to \mathbf{et2e}(\mathcal{ET}')$ is a morphism of **TES** for all morphisms $(\mu,\lambda) : \mathcal{ET} \to \mathcal{ET}'$ of **TET**. Since $(\mu,\lambda) : \mathcal{ET} \to \mathcal{ET}'$ is a morphism of **TET**, we may conclude that $\mu : E \to E'$ and $\lambda : L \to L'$ are functions and $l' \circ \mu = \lambda \circ l$.

Take an arbitrary configuration C in the timed event structure $\mathbf{et2e}(\mathcal{ET})$. Check that $\mu\ C \in \mathbf{C}(\mathbf{et2e}(\mathcal{ET}'))$.

Since C is a configuration, it holds that $C \in Con$ and if $e < e' \in C$ then $e \in C$. Hence, there exist events $e_1, \ldots, e_k \in E$ such that $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ in $\mathcal{ET}$ for some $s_1, \ldots, s_k \in S$ and $C \subseteq \{e_1, \ldots, e_k\}$. According to the item (iii) of Definition 18, we get that $s'_{in} \xrightarrow[eot'_1,\ lot'_1]{\mu(e_1)} s'_1 \ldots s'_{k-1} \xrightarrow[eot'_k,\ lot'_k]{\mu(e_k)} s'_k$ $(k \geq 0)$ in $\mathcal{ET}'$ for some $s'_1, \ldots, s'_k \in S'$ and for some $eot'_1, \ldots, eot'_k, lot'_1, \ldots, lot'_k \in \mathbf{R}$ such that $eot'_j \leq eot_j$ and $lot_j \leq lot'_j$ for all $1 \leq j \leq k$. Thus $\{\mu(e_1), \ldots, \mu(e_k)\} \in Con'$ and $\mu\ C \subseteq \{\mu(e_1), \ldots, \mu(e_k)\}$. Hence, $\mu\ C \in Con'$.

Let $e' \in \mu\ C$ and $e'' <' e'$. This means that $e' = \mu(e_j)$ for some $1 \leq j \leq k$ such that $e_j \in C$. Thus, $e'' \in \mu(e_j) \downarrow$. According to item (i) of definition 18 we have that $\mu(e_j) \downarrow \subseteq \mu(e_j \downarrow)$. Using the fact that C is a configuration, we may conclude that $e'' \in \mu(e_j) \downarrow \subseteq \mu(e_j \downarrow) \subseteq \mu\ C$. Thus, $\mu\ C$ is a configuration.

Now we need to show that $\forall e, e' \in C \diamond$ if $\mu(e) = \mu(e')$ then $e = e'$. Assume that it is not true. Then we have $e, e' \in C$ such that $\mu(e) = \mu(e')$ and $e \neq e'$. This implies that $e = e_j$ and $e' = e_l$ for some $1 \leq j, l \leq k$. W.l.o.g. assume that $j \leq l$. Then there is a sequence $s'_{j-1} \xrightarrow[eot'_j,\ lot'_j]{\mu(e_j)} s'_j \ldots s'_{l-1} \xrightarrow[eot'_l,\ lot'_l]{\mu(e_l)=\mu(e_j)} s'_l$. This contradicts the item (iii) of Definition 17.

Note that $Eot'(\mu(e)) \leq Eot(e)$ and $Lot(e) \leq Lot'(\mu(e))$ for all $e \in E$ due to the item (iv) of Definition 18.

Thus, (μ,λ) is a morphism of **TES** between $\mathbf{et2e}(\mathcal{ET})$ and $\mathbf{et2e}(\mathcal{ET}')$ by Definition 16.

Third, consider an identity morphism $(1_E, 1_L) : \mathcal{ET} \to \mathcal{ET}$ and a pair of morphisms $(\mu,\lambda) : \mathcal{ET} \to \mathcal{ET}'$ and $(\mu',\lambda') : \mathcal{ET}' \to \mathcal{ET}''$ from **TET**. Obviously, $\mathbf{et2e}(1_E, 1_L) = (1_E, 1_L)$, and $\mathbf{et2e}((\mu',\lambda') \circ (\mu,\lambda)) = \mathbf{et2e}(\mu' \circ \mu, \lambda' \circ \lambda) = (\mu' \circ \mu, \lambda' \circ \lambda) = (\mu',\lambda') \circ (\mu,\lambda) = \mathbf{et2e}(\mu',\lambda') \circ \mathbf{et2e}(\mu,\lambda)$. Thus, we have that **et2e** is indeed a functor.

Finally, we should prove that **et2e** is a faithful functor. Take arbitrary objects $\mathcal{ET}$ and $\mathcal{ET}'$ of **TET**. Define a function $F_{\mathcal{ET},\mathcal{ET}'} : \mathbf{TET}(\mathcal{ET},\mathcal{ET}') \to \mathbf{TES}(\mathbf{et2e}(\mathcal{ET}), \mathbf{et2e}(\mathcal{ET}'))$ such that $F_{\mathcal{ET},\mathcal{ET}'}(\mu,\lambda) = \mathbf{et2e}(\mu,\lambda) = (\mu,\lambda)$ for all morphisms $(\mu,\lambda) : \mathcal{ET} \to \mathcal{ET}'$ of **TET**. It is obvious that $F_{\mathcal{ET},\mathcal{ET}'}$ is a function, because **et2e** is a functor. Clearly, $F_{\mathcal{ET},\mathcal{ET}'}$ is injective, because $F_{\mathcal{ET},\mathcal{ET}'}(\mu,\lambda) = (\mu,\lambda)$. Hence, **et2e** is a faithful functor. □

Proposition 8. *Let $\mathcal{E} = (E, <, Con, L, l, Eot, Lot)$ be a timed event structure. Then* $\mathbf{e2et}(\mathcal{E})$ *is a timed event tree,* $(1_E, 1_L) : \mathbf{et2e}(\mathbf{e2et}(\mathcal{E})) \to \mathcal{E}$ *is an isomorphism and the pair* $(\mathbf{e2et}(\mathcal{E}), (1_E, 1_L))$ *is a coreflection of* $\mathcal{E}$ *along* **et2e**.

Доказательство. Obviously, $\mathbf{e2et}(\mathcal{E}) = (S, \epsilon, E, Tran, <, L, l, Eot, Lot)$, where $S = \{e_1 \ldots e_n$

$\in E^* \mid n \geq 0,\ \{e_1,\ldots,e_n\} \in \mathbf{C}(\mathcal{E})$ and for all $1 \leq i,j \leq n$ if $(e_i < e_j)$ then $(i < j)\}$ and $Tran = \{(e_1\ldots e_n,\ e_{n+1},\ Eot(e_{n+1}),\ Lot(e_{n+1}),\ e_1\ldots e_n e_{n+1}) \mid e_1\ldots e_n,\ e_1\ldots e_n e_{n+1} \in S\}$. Moreover, we can easily see that $\mathbf{et2e}(\mathbf{e2et}(\mathcal{E})) = (E,\ <,\ Con*,\ L,\ l,\ Eot,\ Lot)$, where $Con*$ exactly contains all subsets A of events from E such that $A \subseteq \{e_1,\ldots,e_k\}$ and $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ for some $s_1,\ldots,s_k \in S$ and $e_1,\ldots,e_k \in E$. It is easy to check that $Con = Con*$. This implies that $\mathbf{et2e}(\mathbf{e2et}(\mathcal{E})) = \mathcal{E}$. Hence, we have that $(1_E, 1_L) : \mathbf{et2e}(\mathbf{e2et}(\mathcal{E})) = \mathcal{E} \to \mathcal{E}$ is a morphism of **TES** and, moreover, it is an isomorphism.

Finally, show that $(\mathbf{e2et}(\mathcal{E}), (1_E, 1_L))$ is a coreflection of $\mathcal{E}$ along $\mathbf{et2e}$. Consider a timed event tree $\mathcal{ET}' = (S', s'_{in}, E', T', <', L', l', Eot', Lot')$ and a morphism $(\mu,\lambda) : \mathbf{et2e}(\mathcal{ET}') \to \mathcal{E}$ and show that there is a unique morphism $(f,\varsigma) : \mathcal{ET}' \to \mathbf{e2et}(\mathcal{E})$ such that $(\mu,\lambda) = (1_E, 1_L) \circ \mathbf{et2e}(f,\varsigma)$. From this equation, it follows that (f,ς) must match (μ,λ), because $\mathbf{et2e}(f,\varsigma) = (f,\varsigma)$. Hence, we only need to show that (μ,λ) is a morphism of **TET** between $\mathcal{ET}'$ and $\mathbf{e2et}(\mathcal{E})$.

Clearly, $\mathbf{et2e}(\mathcal{ET}') = (E',\ <',\ Con',\ L',\ l',\ Eot',\ Lot')$, where Con' exactly contains all subsets A of events from E such that $A \subseteq \{e_1,\ldots,e_k\}$ and there are states $s_1,\ldots,s_k \in S'$ $(k \geq 0)$ such that $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ $(k \geq 0)$ for some real numbers $eot_1,\ldots,eot_k$, $lot_1,\ldots,lot_k$. Because $(\mu,\lambda) : \mathbf{et2e}(\mathcal{ET}') \to \mathcal{E}$ is a morphism of **TES**, we have that $\mu : E' \to E$ and $\lambda : L' \to L$ are functions, $l \circ \mu = \lambda \circ l'$ and for all $e \in E$ it holds that $Eot(\mu(e)) \leq Eot'(e)$ and $Lot'(e) \leq Lot(\mu(e))$. Prove that (μ,λ) satisfies the other requirements from Definition 18. First, check that $\mu(e)\downarrow \subseteq \mu(e\downarrow)$. Let $e \in E'$. Since $\mathbf{et2e}(\mathcal{ET}')$ is a timed event structure, $e\downarrow \cup \{e\}$ is a configuration. Because $(\mu,\lambda) : et2e(\mathcal{ET}') \to \mathcal{E}$ is a morphism of **TES**, we have that $\mu(e\downarrow \cup \{e\})$ is a configuration too. Hence, $\mu(e)\downarrow \subseteq \mu(e\downarrow \cup \{e\}) = \mu(e\downarrow) \cup \{\mu(e)\}$. Clearly, if $e' \in \mu(e)\downarrow$ then $e' \neq \mu(e)$. Thus, $\mu(e)\downarrow \subseteq \mu(e\downarrow)$.

Next, assume that $s_{in} \xrightarrow[eot_1,\ lot_1]{e_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{e_k} s_k$ for some $k \geq 0$. Hence, $\{e_1,\ldots,e_j\} \in Con'$ for all $1 \leq j \leq k$. Moreover, for all $1 \leq j \leq k$ $\{e_1,\ldots,e_j\}$ is left-closed by Lemma 4. Thus, we get that $\{e_1,\ldots,e_j\} \in \mathbf{C}(et2e(\mathcal{ET}'))$ for all $1 \leq j \leq k$. Since $(\mu,\lambda) : et2e(\mathcal{ET}') \to \mathcal{E}$ is a morphism of **TES** it holds that $\{\mu(e_1),\ldots,\mu(e_j)\} \in \mathbf{C}(\mathcal{E})$ for all $1 \leq j \leq k$. Hence, for all $1 \leq j,l \leq k$ it holds that $\mu(e_j) < \mu(e_l) \Rightarrow j < l$. Let $s'_j = \mu(e_1),\ldots,\mu(e_j)$ for all $1 \leq j \leq k$. According to the definition of **e2et**, we have $s'_1,\ldots,s'_k \in S$ and $\epsilon \xrightarrow[Eot(\mu(e_1)),\ Lot(\mu(e_1))]{\mu(e_1)} s'_1 \ldots s'_{k-1} \xrightarrow[Eot(\mu(e_k)),\ Lot(\mu(e_k))]{\mu(e_k)} s'_k$. Moreover, according to Definition 17 and Definition 18, we have that $Eot(\mu(e_j)) \leq Eot'(e_j) \leq eot_j$ and $lot_j \leq Lot'(e_j) \leq Lot(\mu(e_j))$ for all $1 \leq j \leq k$. □

Using the results mentioned above, we can formulate the following theorem.

Theorem 3. *The functor* **et2e** *is left adjoint to the functor* **e2et** *and this adjunction is a reflection.*

Доказательство. The first part of this theorem follows from Proposition 8 and from the fact that for all morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ it is true that $(1_{E'}, 1_{L'}) \circ \mathbf{et2e}(\mathbf{e2et}(\mu, \lambda)) = (1_{E'}, 1_{L'}) \circ \mathbf{et2e}(\mu, \lambda) = (1_{E'}, 1_{L'}) \circ (\mu, \lambda) = (\mu, \lambda) = (\mu, \lambda) \circ (1_E, 1_L)$. Moreover, due to Proposition 8, we have that the counit η associates each timed event structure $\mathcal{E} = (E, <, Con, L, l, Eot, Lot)$ with the isomorphism $(1_E, 1_L) : \mathbf{et2e}(\mathbf{e2et}(\mathcal{E})) \to \mathcal{E}$. Hence, η is a natural isomorphism. □

Thus, **TES** embeds fully and faithfully into **TET** and is equivalent to the full subcategory of **TET** consisting of those timed event trees $\mathcal{ET}$ that are isomorphic to $\mathbf{e2et}(\mathbf{et2e}(\mathcal{ET}))$.

4.4. A coreflection between the categories TES and TST. It is a well-known fact that there exists a coreflection from the category of synchronization trees to the category of event structures. In this subsection, we try to extend this result to timed variants of the models mentioned above. Clearly, the configurations of a timed event structure can be translated to a tree. Hence, we can specify the following functor $\mathbf{e2s} : \mathbf{TES} \to \mathbf{TST}$.

Definition 24. *Let $\mathcal{E} = (E, <, Con, L, l, Eot, Lot)$ and $\mathcal{E}' = (E', <', Con', L', l', Eot', Lot')$ be timed event structures and $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ be a morphism of* **TES**. *Define $\mathbf{e2s}(\mathcal{E}) = (S, \epsilon, L, Tran)$, where $S = \{e_1 \dots e_n \mid n \geq 0, \{e_1, \dots, e_n\} \in \mathbf{C}(\mathcal{E})$ and for all $1 \leq i, j \leq n$ $(e_i < e_j) \Rightarrow (i < j)\}$ and $Tran = \{(e_1 \dots e_n,\ l(e_{n+1}),\ Eot(e_{n+1}),\ Lot(e_{n+1}),\ e_1 \dots e_n e_{n+1}) \mid e_1 \dots e_n,\ e_1 \dots e_n e_{n+1} \in S\}$, and $\mathbf{e2s}(\mu, \lambda) = (\overline{\mu}, \lambda)$, where $\overline{\mu} : S \to S'$ is defined as: $\overline{\mu}(e_1 \dots e_n) = \mu(e_1) \dots \mu(e_n)$ for all $e_1 \dots e_n \in S$.*

Proposition 9. *The mapping* **e2s** *is a faithful functor.*

Доказательство. First, by the definition of the sets S and $Tran$, we get that $\mathbf{e2s}(\mathcal{E})$ is a timed synchronization tree for all timed event structures $\mathcal{E}$.

Second, we need to prove that $\mathbf{e2s}(\mu, \lambda) : \mathbf{e2s}(\mathcal{E}) \to \mathbf{e2s}(\mathcal{E}')$ is a morphism of **TST** for all morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ of **TES**, where $\mathbf{e2s}(\mu, \lambda) = (\overline{\mu}, \lambda)$ and $\overline{\mu}(e_1 \dots e_n) = \mu(e_1) \dots \mu(e_n)$. Since $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ is a morphism of **TES**, it is easy to see that $\mu : E \to E'$ and $\overline{\mu} : S \to S'$ are functions. Check that the pair $(\overline{\mu}, \lambda)$ satisfies the requirements of Definition 18. It is obvious that $\overline{\mu}(\epsilon) = \epsilon$. Assume that $(e_1 \dots e_{k-1}, l(e_k), Eot(e_k), Lot(e_k), e_1 \dots e_{k-1} e_k) \in Tran$. This means that $s', s'' \in S'$, where $s' = \overline{\mu}(e_1 \dots e_{k-1})$ and $s'' = \overline{\mu}(e_1 \dots e_{k-1} e_k) = s' \mu(e_k)$. By construction of $Tran'$, we have $\overline{\mu}(e_1 \dots e_{k-1}) \xrightarrow[Eot'(\mu(e_k)),\ Lot'(\mu(e_k))]{l'(\mu(e_k))} \overline{\mu}(e_1 \dots e_{k-1} e_k))$. Moreover, $l' \circ \mu(e_k) = \lambda \circ l(e_k)$ and $Eot'(\mu(e_k)) \leq Eot(e_k)$ and $Lot(e_k) \leq Lot'(\mu(e_k))$, since (μ, λ) is a morphism of **TES**. Thus, $(\overline{\mu}, \lambda)$ is indeed a morphism of **TST** from $\mathbf{e2s}(\mathcal{E})$ to $\mathbf{e2s}(\mathcal{E}')$.

Third, we should contemplate an identity morphism $(1_E, 1_L) : \mathcal{E} \to \mathcal{E}$ and two morphisms $(\mu, \lambda) : \mathcal{E} \to \mathcal{E}'$ and $(\mu', \lambda') : \mathcal{E}' \to \mathcal{E}''$ from **TES**. Obviously, $\mathbf{e2s}(1_E, 1_L) = (\overline{1_E}, 1_L) = (1_S, 1_L)$, and $\mathbf{e2s}((\mu', \lambda') \circ (\mu, \lambda)) = \mathbf{e2s}(\mu' \circ \mu, \lambda' \circ \lambda) = (\overline{\mu' \circ \mu}, \lambda' \circ \lambda) = (\overline{\mu'}, \lambda') \circ (\overline{\mu}, \lambda) = \mathbf{e2s}(\mu', \lambda') \circ \mathbf{e2s}(\mu, \lambda)$. Thus, we have that **e2s** is indeed a functor.

Finally, we need to clarify that **e2s** is a faithful functor. Take an arbitrary timed event structures $\mathcal{E}$ and $\mathcal{E}'$ from **TES**. Specify a function $F_{\mathcal{E},\mathcal{E}'} : \mathbf{TES}(\mathcal{E},\mathcal{E}') \to \mathbf{TST}(\mathbf{e2s}(\mathcal{E}),\mathbf{e2s}(\mathcal{E}'))$ such that $F_{\mathcal{E},\mathcal{E}'}(\mu,\lambda) = \mathbf{e2s}(\mu,\lambda) = (\overline{\mu},\lambda)$ for all morphisms $(\mu,\lambda) : \mathcal{E} \to \mathcal{E}'$ of **TES**. Because **e2s** is a functor, we have that $F_{\mathcal{E},\mathcal{E}'}$ is a function.

Next, we need to verify that $F_{\mathcal{E},\mathcal{E}'}$ is an injective function. Take two arbitrary morphisms $(\mu_1,\lambda_1) : \mathcal{E} \to \mathcal{E}'$ and $(\mu_2,\lambda_2) : \mathcal{E} \to \mathcal{E}'$ such that $F_{\mathcal{E},\mathcal{E}'}(\mu_1,\lambda_1) = F_{\mathcal{E},\mathcal{E}'}(\mu_2,\lambda_2)$. This implies that $(\overline{\mu_1},\lambda_1) = (\overline{\mu_2},\lambda_2)$. Hence, $\lambda_1 = \lambda_2$ and $\overline{\mu_1} = \overline{\mu_2}$. Take an arbitrary event $e \in E$. Since $\mathcal{E}$ is a timed event structure, we have a configuration $\{e_1,\dots,e_n\} = e\downarrow \cup \{e\}$ of $\mathcal{E}$ such that $e_n = e$ and for all $1 \le i,j \le n$ if $e_i < e_j$ then $i < j$. This implies that $s_i = e_1 \dots e_i \in S$ for all $1 \le i \le n$. Since $\overline{\mu_1} = \overline{\mu_2}$, we have that $\mu_1(e_i) = \mu_2(e_i)$ for all $1 \le i \le n$. Hence, $\mu_1 = \mu_2$. Thus, $F_{\mathcal{E},\mathcal{E}'}$ is injective, i.e. **e2s** is a faithful functor. □

Next, we try to transform a timed synchronization tree $\mathcal{S}$ into some timed event structure $\mathcal{E}$, assuming that each transition of $\mathcal{S}$ represents a separate event with the same timed limits as this transition, defining the set of consistent events as a set of transitions that appear together on some branch and specifying the causal dependency relation as the hierarchy of transitions in the tree structure. Thus, we can specify a functor $\mathbf{s2e} : \mathbf{TST} \to \mathbf{TES}$.

Definition 25. *Let $\mathcal{S} = (S, s_{in}, L, T)$ and $\mathcal{S}' = (S', s'_{in}, L', T')$ be timed synchronization trees and $(\sigma,\lambda) : \mathcal{S} \to \mathcal{S}'$ be a morphism of* **TST**. *Define* $\mathbf{s2e}(\mathcal{S}) = (T,\ <^*,\ Con^*,\ L,\ l^*,\ Eot^*,\ Lot^*)$, *where $(s,a,eot,lot,s') <^* (u,b,eot',lot',u') \iff$ there is a sequence $s' \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = u$ for some $k \ge 1$, $Con^* = \{A \subseteq \{t_1,\dots,tr_k\} \mid tr_1 = (s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1),\ \dots,\ tr_k = (s_{k-1},\ a_k,\ eot_k,\ lot_k,\ s_k) \in T,\ (k \ge 0)\}$, $l^*(s,a,eot,lot,s') = a$, $Eot^*(s,a,eot,lot,s') = eot$ and $Lot^*(s,a,eot,lot,s') = lot$. Moreover, define* $\mathbf{s2e}(\sigma,\lambda) = (\mu,\lambda)$, *where $\mu(s,a,eot,lot,s') = (\sigma(s),\lambda(a),eot',lot',\sigma(s'))$ for some $eot',lot' \in \mathbf{R}$.*

Lemma 7. *For any timed synchronization tree $\mathcal{S}$, if $C \in \mathbf{C}(\mathbf{s2e}(\mathcal{S}))$ then $C = \{(s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1),\ \dots,\ (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n) \mid s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \dots s_{n-1} \xrightarrow[eot_n,\ lot_n]{a_n} s_n\}$ for some $n \ge 0$.*

Proposition 10. *The mapping* **s2e** *is a faithful functor.*

Доказательство. First, we need to show that $\mathbf{s2e}(\mathcal{S})$ is a timed event structure for all timed synchronization trees $\mathcal{S}$. It is easy to check that $<^*$ is a strict order and, for all $(s,a,eot,lot,s') \in T$, $Eot^*(s,a,eot,lot,s') \le Lot^*(s,a,eot,lot,s')$. Take an arbitrary $(s,a,eot,lot,s') \in T$. Now we need to verify that $(s,a,eot,lot,s')\downarrow = \{(u,b,eot',lot',u') \in T \mid (u,\ b,\ eot',\ lot',\ u') <^* (s,\ a,\ eot,\ lot,\ s')\}$ is a finite set. Since $\mathcal{S}$ is a timed synchronization tree, we can find a unique sequence $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \dots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = s$ $(k \ge 0)$. Using the definition of a timed synchronization tree, we get that $(s,\ a,\ eot,\ lot,\ s')\downarrow \subseteq \{(s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1),\ \dots,\ (s_{k-1},\ a_k,\ eot_k,\ lot_k,\ s_k)\}$. Hence, $(s,a,eot,lot,s')\downarrow$ is a finite set. Moreover, it immediately follows from the definition of

Con^* that $\forall (s,a,eot,lot,s') \in T \diamond \{(s,a,eot,lot,s')\} \in Con^*$ and $Y \subseteq X \in Con^* \Rightarrow Y \in Con^*$. By the definition of the relation $<^*$, we get that for all $X \in Con^*$, if $(u,b,eot',lot',u') <^* (s,a,eot,lot,s') \in X$ then $X \cup \{(u,\ b,\ eot',\ lot',\ u')\} \in Con^*$.

Second, we have to prove that $\mathbf{s2e}(\sigma,\lambda)$ is a morphism of **TES**. Assume that $\mathbf{s2e}(\mathcal{S}) = (T, <^*, Con^*, L, l^*, Eot^*, Lot^*)$ and $\mathbf{s2e}(\mathcal{S}') = (T', <'^*, Con'^*, L', l'^*, Eot'^*, Lot'^*)$. Since $(\sigma,\lambda) : \mathcal{S} \to \mathcal{S}'$ is a morphism of **TST**, we may conclude that $\lambda : L \to L'$ and $\sigma : S \to S'$ are functions, $\sigma(s_{in}) = s'_{in}$ and for each $(s,a,eot,lot,s') \in T$ there is the only $(\sigma(s),\lambda(a),eot',lot',\sigma(s')) \in T'$, where $eot' \le eot$ and $lot \le lot'$. Hence, μ, defined as $\mu(s,a,eot,lot,s') = (\sigma(s),\ \lambda(a),\ eot',\ lot',\ \sigma(s'))$, is a function. Take an arbitrary configuration C of $\mathbf{s2e}(\mathcal{S})$ and check that $\mu\ C \in \mathbf{C}(\mathbf{s2e}(\mathcal{S}'))$. Since C is a configuration, it holds that $C \in Con^*$ and if $e <^* e' \in C$ then $e \in C$. Hence, there exist states $s_1,\ \ldots,\ s_k \in S$ $(k \ge 0)$ such that $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k$ and $C \subseteq \{(s_0,a_1,eot_1,lot_1,s_1), \ldots, (s_{k-1},a_k,eot_k,lot_k,s_k)\}$. W.l.o.g. suppose $e_i = (s_{i-1},a_i,eot_i,lot_i,s_i)$ $(i = 1,\ldots,k)$, where $s_0 = s_{in}$. Because (σ,λ) is a morphism of **TST**, we get that $\sigma(s_{in}) \xrightarrow[eot'_1,\ lot'_1]{\lambda(a_1)} \sigma(s_1) \ldots \sigma(s_{k-1}) \xrightarrow[eot'_k,\ lot'_k]{\lambda(a_k)} \sigma(s_k)$ in $\mathcal{S}'$. Moreover, it is easy to see that $\mu(e_i) = (\sigma(s_{i-1}),\lambda(a_i),eot'_i,lot'_i,\sigma(s_i))$ for all $1 \le i \le k$. Thus, $\mu\ C \subseteq \{\mu(e_1),\ \ldots,\ \mu(e_k)\}$ and $\{\mu(e_1),\ \ldots,\ \mu(e_k)\} \in Con'^*$. Clearly, for all $e_i \in C$, $\mu(e_i)\downarrow = \{\mu(e_1),\ \ldots,\ \mu(e_{i-1})\}$. Hence, $\mu\ C$ is a configuration. Note that $\forall e_i,\ e_j \in C \diamond$ if $\mu(e_i) = \mu(e_j)$ then $i = j$. Moreover, it is obvious that $Eot'^*(\mu(s,\ a,\ eot,\ lot,\ s')) \le Eot^*(s,\ a,\ eot,\ lot,\ s')$ and $Lot^*(s,\ a,\ eot,\ lot,\ s') \le Lot'^*(\mu(s,\ a,\ eot,\ lot,\ s'))$ for all $(s,a,eot,lot,s') \in T$, since $\mu(s,a,eot,lot,s') = (\sigma(s),\lambda(a),eot',lot',\sigma(s'))$ with $eot' \le eot$ and $lot \le lot'$. Thus, (μ,λ) is indeed a morphism of **TES**.

Third, we should contemplate an identity morphism $(1_S,1_L) : \mathcal{S} \to \mathcal{S}$ and two morphisms $(\sigma,\lambda) : \mathcal{S} \to \mathcal{S}'$ and $(\sigma',\lambda') : \mathcal{S}' \to \mathcal{S}''$ from **TST**. Obviously, $\mathbf{s2e}(1_S,1_L) = (\mu_{1_E,1_L},1_L) = (1_E,1_L)$, and $\mathbf{s2e}((\sigma',\lambda')\circ(\sigma,\lambda)) = \mathbf{s2e}(\sigma'\circ\sigma,\lambda'\circ\lambda) = (\mu_{\sigma'\circ\sigma,\lambda'\circ\lambda},\lambda'\circ\lambda) = (\mu_{\sigma',\lambda'},\lambda')\circ(\mu_{\sigma,\lambda},\lambda) = \mathbf{s2e}(\sigma',\lambda')\circ\mathbf{s2e}(\sigma,\lambda)$. Hence, **s2e** is a functor.

Finally, show that **s2e** is a fully faithful functor. Take arbitrary timed synchronization trees $\mathcal{S}$ and $\mathcal{S}'$ from **TST**. Define a mapping $F_{\mathcal{S},\mathcal{S}'} : \mathbf{TST}(\mathcal{S},\mathcal{S}') \to \mathbf{TES}(\mathbf{s2e}(\mathcal{S}),\mathbf{s2e}(\mathcal{S}'))$ such that $F_{\mathcal{S},\mathcal{S}'}(\sigma,\lambda) = \mathbf{s2e}(\sigma,\lambda) = (\mu_{\sigma,\lambda},\lambda)$ for all morphisms $(\sigma,\lambda) : \mathcal{S} \to \mathcal{S}'$ of **TST**. Because **s2e** is a functor, we have that $F_{\mathcal{S},\mathcal{S}'}$ is a function.

Check that $F_{\mathcal{S},\mathcal{S}'}$ is a bijective function. Take arbitrary morphisms $(\sigma_1,\lambda_1) : \mathcal{S} \to \mathcal{S}'$ and $(\sigma_2,\lambda_2) : \mathcal{S} \to \mathcal{S}'$ such that $F_{\mathcal{S},\mathcal{S}'}(\sigma_1,\lambda_1) = F_{\mathcal{S},\mathcal{S}'}(\sigma_2,\lambda_2)$. This means that $(\mu_{\sigma_1,\lambda_1},\lambda_1) = (\mu_{\sigma_2,\lambda_2},\lambda_2)$. Hence, $\lambda_1 = \lambda_2$ and $\mu_{\sigma_1,\lambda_1} = \mu_{\sigma_2,\lambda_2}$. Take an arbitrary state $s \in S$. Because $\mathcal{S}$ is a timed synchronization tree, we can find the only transition $(s',a,eot,lot,s) \in T$. Since $\mu_{\sigma_1,\lambda_1} = \mu_{\sigma_2,\lambda_2}$, we have $(\sigma_1(s'),\lambda_1(a),eot',lot',\sigma_1(s)) = (\sigma_2(s'),\lambda_2(a) = \lambda_1(a),eot'',lot'',\sigma_2(s))$. This implies that $\sigma_1(s) = \sigma_2(s)$. Hence, $F_{\mathcal{S},\mathcal{S}'}$ is injective, i.e. **s2e** is a faithful functor. Next, take an arbitrary morphism $(\mu,\lambda) : \mathbf{s2e}(\mathcal{S}) \to \mathbf{s2e}(\mathcal{S}')$ of **TES**. Define a function $g : S \to S'$ as follows: $g(s_{in}) = s'_{in}$

and for all $s \in S$ such that $s \neq s_{in}$, $g(s) = last(\mu(tr_s))$, where $tr_s \in T$ such that $last(tr_s) = s$ and $last$ is a function which maps each transition to it's last state, i.e. $last(u, b, eot', lot', u') = u'$ for all transitions (u, b, eot', lot', u'). Since $\mathcal{S}$ is a timed synchronization tree, there is the only transition $tr_s \in T$ with $last(tr_s) = s$. This means that g is indeed a function. Since $\mu(tr)\downarrow = \mu(tr\downarrow)$ for all $tr \in T$ and $l'^* \circ \mu = l^*$, we get that $(last(\mu(tr_s)), \lambda(a), eot', lot', last(\mu(tr_{s'}))) = \mu(tr_{s'}) \in T'$, where $tr_{s'} = (s, a, eot, lot, s')$. This implies that (g, λ) is a morphism of **TST** from $\mathcal{S}$ to $\mathcal{S}'$ and $F_{\mathcal{S},\mathcal{S}'}(g, \lambda) = (\mu, \lambda)$. Thus, **s2e** is a full functor. □

Proposition 11. *Let $\mathcal{S} = (S,\ s_{in},\ L,\ T)$ be a timed synchronization tree. Then there is an isomorphism $(\eta^*_S, 1_L) : \mathcal{S} \to$ **e2s**(**s2e**($\mathcal{S}$)) such that the pair (**s2e**($\mathcal{S}$), $(\eta^*, 1_L)$) is a reflection of $\mathcal{S}$ along **e2s**.*

Доказательство. Note, that **s2e**($\mathcal{S}$) $= (T,\ <^*,\ Con^*,\ L,\ l^*,\ Eot^*,\ Lot^*)$, where $<^*$, Con^*, l^*, Eot^* and Lot^* are defined as in Definition 25. Furthermore, **e2s**(**s2e**($\mathcal{S}$)) $= (S^*,\ \epsilon,\ L,\ Tran)$, where $S^* = \{(s_0,\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n) \mid n \geq 0,\ \{(s_0,\ a_1,\ eot_1,\ lot_1,\ s_1), \ldots, (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n)\} \in$ **C**(**s2e**($\mathcal{S}$)) and for all $1 \leq i, j \leq n$ $(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i) <^* (s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j) \Rightarrow (i < j)\}$ and $Tran = \{((s_0,\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n),\ l^*((s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1})),\ Eot^*((s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1})),\ Lot^*((s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1})),\ (s_0,\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n)\ (s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1})) \mid (s_0,\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n),\ (s_0,\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n)\ (s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1}) \in S^*\}$.

By Lemma 7 we may conclude that $S^* = \{(s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n) \mid n \geq 0$ and $(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i) \in T$ for all $1 \leq i \leq n\}$ and $Tran = \{((s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n),\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ (s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n)\ (s_n,\ a_{n+1},\ eot_{n+1},\ lot_{n+1},\ s_{n+1})) \mid (s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i) \in T$ for all $1 \leq i \leq n+1\}$.

Define a mapping $\eta^*_S : S \to S^*$ as follows: for all $s \in S \diamond \eta^*_S(s) = (s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{k-1},\ a_k,\ eot_k,\ lot_k,\ s_k))$, where $s_k = s$. It is easy to see that for all $s \in S$ there is a unique sequence $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = s$ with $k \geq 0$ by Definition 12. Hence, η^*_S is a function and $\eta^*_S(s_{in}) = \epsilon$.

Define a mapping $\eta^{**}_S : S^* \to S$ as follows: for all $s^* \in S^* \diamond \eta^{**}_S(s^*) = \eta^{**}_S((s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{k-1}, a_k,\ eot_k,\ lot_k,\ s_k)) = s_k$ and $\eta^{**}_S(\epsilon) = s_{in}$. Clearly, η^{**}_S is a function and $\eta^{**}_S \circ \eta^*_S = 1_S$ and $\eta^*_S \circ \eta^{**}_S = 1_{S^*}$.

Now, we need to prove that $(\eta^*_S, 1_L) : \mathcal{S} \to$ **e2s**(**s2e**($\mathcal{S}$)) is a morphism of **TST**. Obviously, η^*_S and 1_L are functions and $\eta^*_S(s_{in}) = \epsilon$. Take an arbitrary $(s, a, eot, lot, s') \in T$. According to Definition 12, we have a unique sequence $s_{in} \xrightarrow[eot_1,\ lot_1]{a_1} s_1 \ldots s_{k-1} \xrightarrow[eot_k,\ lot_k]{a_k} s_k = s$. Hence, $\eta^*_S(s) = (s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{k-1},\ a_k,\ eot_k,\ lot_k,\ s_k)$, $\eta^*_S(s') = (s_{in},\ a_1,\ eot_1,\ lot_1,\ s_1) \ldots (s_{k-1},\ a_k,\ eot_k,\ lot_k,\ s_k)\ (s,\ a,\ eot,\ lot,\ s')$ and $(\eta^*_S(s),\ a,\ eot,\ lot,\ \eta^*_S(s')) \in Tran$. Thus,

$(\eta^*_S, 1_L) : \mathcal{S} \to \mathbf{e2s}(\mathbf{s2e}(\mathcal{S}))$ is really a morphism of **TST**.

Next, we need to show that $(\eta^{**}_S, 1_L) : \mathbf{e2s}(\mathbf{e2c}(\mathcal{S})) \to \mathcal{S}$ is a morphism of **TST**. Obviously, η^{**}_S is a function and $\eta^{**}_S(\epsilon) = s_{in}$. Suppose that $(t, a, eot, lot, t') \in Tran$ for some $t, t' \in S^*$. This means that $t = (s_{in}, a_1, eot_1, lot_1, s_1) \ldots (s_{k-1}, a_k, eot_k, lot_k, s_k)$, $t' = (s_{in}, a_1, eot_1, lot_1, s_1) \ldots (s_{k-1}, a_k, eot_k, lot_k, s_k)\ (s_k, a, eot, lot, s_{k+1})$ and $(s_k, a, eot, lot, s_{k+1}) \in T$. Since $\eta^{**}_S(t) = s_k$ and $\eta^{**}_S(t') = s_{k+1}$, we may conclude that $(\eta^{**}_S, 1_L) : \mathbf{e2s}(\mathbf{s2e}(\mathcal{S})) \to \mathcal{S}$ is indeed a morphism of **TST**.

Hence, $(\eta^*_S, 1_L)$ and $(\eta^{**}_S, 1_L)$ are morphisms of **TST** and $(\eta^{**}_S, 1_L) \circ (\eta^*_S, 1_L) = (1_S, 1_L)$ and $(\eta^*_S, 1_L) \circ (\eta^{**}_S, 1_L) = (1_{S^*}, 1_L)$. Thus, $(\eta^*_S, 1_L)$ is an isomorphism.

Finally, check that $(\mathbf{s2c}(\mathcal{S}), (\eta^*_S, 1_L))$ is a reflection of $\mathcal{S}$ along **e2s**, i.e. whenever $\mathcal{E}'$ is a timed event structure and $(\sigma, \lambda) : \mathcal{S} \to \mathbf{e2s}(\mathcal{E}')$ is a morphism of **TST**, there exists a unique morphism $(g, \lambda') : \mathbf{s2e}(\mathcal{S}) \to \mathcal{E}'$ such that $(\sigma, \lambda) = \mathbf{e2s}(g, \lambda') \circ (\eta^*_S, 1_L)$. Since $\mathbf{e2s}(g, \lambda') = (\overline{g}, \lambda')$, we may conclude that λ' must be equal to λ and $\overline{g} \circ \eta^*_S$ must match σ.

Take an arbitrary timed event structure $\mathcal{E}' = (E', <', Con', L', l', Eot', Lot')$ and an arbitrary morphism $(\sigma, \lambda) : \mathcal{S} \to \mathbf{e2s}(\mathcal{E}')$ of **TST**. It is obvious that $\mathbf{e2s}(\mathcal{E}') = (S', \epsilon, L', Tran')$, where $S' = \{e_1 \ldots e_n \mid n \geq 0,\ \{e_1, \ldots, e_n\} \in \mathbf{C}(\mathcal{E})$ and for all $1 \leq i, j \leq n\ (e_i < e_j) \Rightarrow (i < j)\}$ and $Tran' = \{(e_1 \ldots e_n,\ l'(e_{n+1}),\ Eot'(e_{n+1}),\ Lot'(e_{n+1}),\ e_1 \ldots e_n e_{n+1}) \mid e_1 \ldots e_n,\ e_1 \ldots e_n e_{n+1} \in S'\}$. By definition of morphism of **TST**, it holds that $\sigma : S \to S'$ and $\lambda : L \to L'$ are functions, $\sigma(s_{in}) = \epsilon$ and for all $(s, a, eot, lot, s') \in T$ there exist $eot', lot' \in \mathbf{R}$ such that $eot' \leq eot$, $lot \leq lot'$ and $(\sigma(s), \lambda(a), eot', lot', \sigma(s')) \in Tran'$.

Clearly, $\overline{g} \circ \eta^*_S = \sigma \iff$ for all $(s, e, eot, lot, s') \in T$ $g(s, e, eot, lot, s') = e_k$, where $\sigma(s') = e_1 \ldots e_k$ for some $k \geq 0$. We should only show that $(g, \lambda) : \mathbf{s2e}(\mathcal{S}) \to \mathcal{E}'$ is a morphism of **TES**, where $g(s, e, eot, lot, s') = e_k$ with $\sigma(s') = e_1 \ldots e_k$ for some $k \geq 0$. Note that g is a function, because σ is a function. Moreover, for all $(s, a, eot, lot, s') \in T$ there exist $eot', lot' \in \mathbf{R}$ such that $eot' \leq eot$, $lot \leq lot'$ and $(\sigma(s), \lambda(a), eot', lot', \sigma(s')) \in Tran'$. By the definition of $\mathbf{e2s}(\mathcal{E}')$, we have $\sigma(s') = \sigma(s)\ e_k$ and $l'(e_k) = \lambda(a)$. This implies that $l' \circ g(s, a, eot, lot, s') = l'(e_k) = \lambda(a) = \lambda \circ l^*(s, a, eot, lot, s')$ for all $(s, a, eot, lot, s') \in T$.

Assume that $C \in \mathbf{C}(\mathbf{s2e}(\mathcal{S}))$. By Lemma 7 we have that $C = \{(s_{in} = s_0,\ a_1,\ eot_1,\ lot_1,\ s_1), \ldots, (s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n) \mid (s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j) \in T$ for all $1 \leq j \leq n\}$. It is easy to see that $\sigma(s_0) = \epsilon$, $\sigma(s_1) = e'_1$, $\ldots$, $\sigma(s_n) = e'_1\ \ldots\ e'_n$ for some $e'_1, \ \ldots,\ e'_n \in E'$. Hence, $g\ C = \{g(s_{in} = s_0,\ a_1,\ eot_1,\ lot_1,\ s_1), \ldots, g(s_{n-1},\ a_n,\ eot_n,\ lot_n,\ s_n) \mid (s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j) \in T$ for all $1 \leq j \leq n\}$ $= \{e'_1, \ldots, e'_n \mid \sigma(s_n) = e'_1\ \ldots\ e'_n\}$. Thus, $g\ C \in \mathbf{C}(\mathcal{E}')$. Next, consider two transitions $(s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j)$ and $(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i)$ from C. If $g(s_{j-1},\ a_j,\ eot_j,\ lot_j,\ s_j) = g(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i)$ then $e'_i = e'_j$. Hence, $i = j$.

Furthermore, for all $(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i) \in C$ it holds that $Eot'(g(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i)) = Eot'(e'_i) \leq eot_i = Eot^*(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i)$ and $Lot^*(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i) = lot_i \leq Lot'(e'_i) =$

$Lot'(g(s_{i-1},\ a_i,\ eot_i,\ lot_i,\ s_i))$. Thus, (g, λ) is indeed a morphism of **TES** between $\mathbf{s2e}(\mathcal{S})$ and $\mathcal{E}'$. Therefore, $(\mathbf{s2e}(\mathcal{S}), (\eta^*, 1_L))$ is a reflection of $\mathcal{S}$ along **e2s**. □

As a result, the following statement is true.

Theorem 4. *The functor* **e2s** *is right adjoint to* **s2e** *and the adjunction is a coreflection.*

Доказательство. The first statement follows from Proposition 11 and from the fact that for all morphisms $(\sigma, \lambda) : \mathcal{S} = (S,\ s_{in},\ L,\ T) \to \mathcal{S}' = (S',\ s'_{in},\ L',\ T')$ it is true that $(\eta^*_{S'}, 1_{L'}) \circ (\sigma, \lambda) = \mathbf{e2s}(\mathbf{s2e}(\sigma, \lambda)) \circ (\eta^*_S, 1_L)$. Next, due to Lemma 11 we may conclude that the unit ψ associates each timed synchronization tree $\mathcal{S} = (S,\ s_{in},\ L,\ T)$ with the isomorphism $(\eta^*_S, 1_L) : \mathcal{S} \to \mathbf{e2s}(\mathbf{s2e}(\mathcal{S}))$. Hence, ψ is a natural isomorphism. □

Thus, **TST** embeds fully and faithfully into **TES** and is equivalent to the full subcategory of **TES** consisting of those timed event structures $\mathcal{E}$ that are isomorphic to $\mathbf{s2e}(\mathbf{e2s}(\mathcal{E}))$.

4.5. Summary. The following diagram summarizes the functors which relate the models under consideration. Here the hooks represent embeddings and the small triangles between arrows indicate the direction of left adjoints.

$$\begin{array}{ccc} \mathbf{TCT} & \overset{\hookrightarrow}{\underset{\leftarrow}{\triangleright}} & \mathbf{TET} \\ \uparrow\!\vartriangle\!\downarrow & & \uparrow\!\triangledown\!\downarrow \\ \mathbf{TST} & \overset{\hookrightarrow}{\underset{\leftarrow}{\triangleright}} & \mathbf{TES} \end{array}$$

The diagram can be seen as a decomposition of the coreflection from **TST** to **TES** into three consecutive adjunctions. Moreover, it is clear that the embeddings and left adjoints commute. Thus we have derived a composed adjunction between timed causal trees and timed event structures. It is not a coreflection, but it is induced by a coreflection and a reflection via a larger category, **TET**. The object component of the right adjoint of this adjunction amounts to the following transformation: it ‘linearizes’ a timed event structure into a timed causal tree by forgetting about events.

5. Conclusion. In this paper we established some relations between the timed extension of the well-known concurrent models. In particular, we showed that:

- The category of timed synchronization trees embeds fully and faithfully into the category of timed event structures and into the category of timed causal trees.
- There is an adjunction between the category of timed causal trees and the category of timed event structures. This adjunction is represented as the composition of a coreflection from the category of timed causal trees to the category of timed event trees and a reflection from the category of timed event trees to the category of timed event structures.

Thus, as in the case of timeless models, timed causal trees are more trivial than timed event structures because they apply causality without the notion of an event and, at the same time, are more expressive than the latter, because their possible runs can be defined in terms of a tree without restrictions, but the set of possible runs of any event structure must be closed under the shuffling of concurrent transitions.

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Название: Сравнение причинной зависимости и семантики истинного параллелизма в контексте временных моделей

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Аннотация: Цель данной работы — установить взаимосвязи между различными параллельными моделями реального времени. Для достижения данной цели мы определили категорию временных причинных деревьев и исследовали, какое место занимает эта категория среди других категорий временных моделей. В частности, мы установили существование сопряженных функторов между категорией временных причинных деревьев и категорией временных структур событий, используя для этого более выразительную модель временных деревьев событий. Тем самым мы показали, что временные причинные деревья проще временных структур событий в том, что они отражают только один аспект семантики истинного параллелизма, а именно причинную зависимость, и не используют понятие события для задания отношения причинной зависимости. С другой стороны, модель временных причинных деревьев более выразительна, чем модель временных структур событий по следующей причине: для нее множество всех возможных последовательностей выполнения может быть определено в терминах дерева без каких-либо ограничений,

а множество всевозможных последовательностей выполнения для временной структуры событий должно быть замкнутыми относительно операции перестановки параллельных переходов.

Ключевые слова: модели реального времени, истинный параллелизм, причинная зависимость, отношения, унификация, теория категорий